MATH 178 SOLUTIONS - Sec. 7.6

(1) $f(x) = e^{2x}$, $a=0$

$L(x) = f(a) + f'(a)(x-a) = \boxed{1+2x}$ (since $f'(x) = 2e^{2x}$, so $f'(0)=2$)

(3) $f(x) = \frac{1}{1-x}$, $a=0$

$L(x) = f(a) + f'(a)(x-a) = \boxed{1+x}$ (since $f'(x) = -(1-x)^{-2}(-1) = \frac{1}{(1-x)^2}$, so $f'(0)=1$)

(6) $f(x) = \frac{1}{1+x} = (1+x)^{-1}$, $a=0$

$f'(x) = -(1+x)^{-2}$

$f''(x) = 2(1+x)^{-3}$

$f'''(x) = -6(1+x)^{-4}$

$f^{(4)}(x) = 24(1+x)^{-5}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	1
1	-1	-1
2	2	1
3	-6	-1
4	24	1

$$\boxed{P_4(x) = 1 - x + x^2 - x^3 + x^4}$$

(7) $f(x) = \cos x$, $a=0$

$f'(x) = -\sin x$

$f''(x) = -\cos x$

$f'''(x) = \sin x$

$f^{(4)}(x) = \cos x$

$f^{(5)}(x) = -\sin x$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	1
1	0	0
2	-1	$-1/2!$
3	0	0
4	1	$1/4!$
5	0	0

$$\boxed{P_5(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!}} = \boxed{1 - \frac{x^2}{2} + \frac{x^4}{24}}$$

(8) $f(x) = e^{3x}$, $a=0$

$f'(x) = e^{3x}\cdot 3$

$f''(x) = e^{3x}\cdot 3^2$

$f'''(x) = e^{3x}\cdot 3^3$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	1
1	3	3
2	9	$9/2$
3	27	$9/2$

$$\boxed{P_3(x) = 1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3}$$

(10) $f(x) = \sqrt{1+x} = (1+x)^{1/2}$, $a=0$

$f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$f''(x) = -\frac{1}{4}(1+x)^{-3/2}$

$f'''(x) = \frac{3}{8}(1+x)^{-5/2}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	1
1	$1/2$	$1/2$
2	$-1/4$	$-1/8$
3	$3/8$	$1/16$

$$\boxed{P_3(x) = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3}$$

(14) $f(x) = e^{-x}$, $a=0$; $x = .3$

$f'(x) = -e^{-x}$

$f''(x) = e^{-x}$

$f'''(x) = -e^{-x}$

$f^{(4)}(x) = e^{-x}$

$f^{(5)}(x) = -e^{-x}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	1
1	-1	-1
2	1	$1/2$
3	-1	$-1/6$
4	1	$1/24$
5	-1	$-\frac{1}{120}$

$$\boxed{P_5(x) = 1 - x + \frac{x^2}{2} - \frac{x^3}{6} + \frac{x^4}{24} - \frac{x^5}{120}}$$

$e^{-.3} \approx .740818$ AND $P_5(.3) \approx .740817$

(16) $f(x) = \ln(1+x)$, $a = 0$; $x = .1$

$f'(x) = \frac{1}{1+x} = (1+x)^{-1}$

$f''(x) = -(1+x)^{-2}$

$f'''(x) = 2(1+x)^{-3}$

$\boxed{P_3(x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	0	0
1	1	1
2	-1	$-1/2$
3	2	$1/3$

$\ln 1.1 \approx \underline{.09531}$ AND $P_3(1.1) \approx \underline{.09533}$

(17) $f(x) = \sin x$, $a = 0$

a) $f'(x) = \cos x$

$f''(x) = -\sin x$

$f'''(x) = -\cos x$

$\boxed{P_3(x) = x - \frac{x^3}{3!} = x - \frac{x^3}{6}}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	0	0
1	1	1
2	0	0
3	-1	$-\frac{1}{6}$

b) SINCE $\sin x \approx x - \frac{x^3}{6}$ FOR $x \approx 0$, $\frac{\sin x}{x} \approx 1 - \frac{x^2}{6}$ FOR $x \approx 0$;

SO IT IS REASONABLE TO EXPECT THAT $\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \left(1 - \frac{x^2}{6}\right) = \underline{1}$.

(19) $f(x) = \sqrt{x}$, $a = 1$; $x = 2$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$

$f'''(x) = \frac{3}{8}x^{-5/2}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	1
1	$1/2$	$1/2$
2	$-1/4$	$-1/8$
3	$3/8$	$1/16$

$\boxed{P_3(x) = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \frac{1}{16}(x-1)^3}$

$\sqrt{2} \approx \underline{1.4142}$ AND $P_3(2) = \underline{1.4375}$

REMARK WE COULD ALSO FIND $P_3(x)$ BY SUBSTITUTING $x-1$ FOR x IN THE ANSWER TO #10.

(20) $f(x) = \ln x$, $a = 1$; $x = 2$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	0	0
1	1	1
2	-1	$-1/2$
3	2	$1/3$

$\boxed{P_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3}$

$\ln 2 \approx \underline{.693}$ AND $P_3(2) \approx \underline{.833}$

REMARK WE COULD ALSO FIND $P_3(x)$ BY SUBSTITUTING $x-1$ FOR x IN THE ANSWER TO #16.

(23) $f(x) = e^x$, $a = 2$; $x = 2.1$

$f'(x) = e^x$

$f''(x) = e^x$

$f'''(x) = e^x$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	e^2	e^2
1	e^2	e^2
2	e^2	$e^2/2$
3	e^2	$e^2/6$

$\boxed{P_3(x) = e^2 + e^2(x-2) + \frac{e^2}{2}(x-2)^2 + \frac{e^2}{6}(x-2)^3}$

$= \boxed{e^2\left[1 + (x-2) + \frac{1}{2}(x-2)^2 + \frac{1}{6}(x-2)^3\right]}$

$e^{2.1} \approx \underline{8.16617}$ AND $P_3(2.1) \approx \underline{8.16614}$

REMARK WE COULD ALSO FIND $P_3(x)$ USING $e^x = e^2 \cdot e^{x-2}$ AND THEN SUBSTITUTING $x-2$ IN FOR x IN THE TAYLOR POLYNOMIAL $P_3(x)$ FOR $f(x) = e^x$ AT $a = 0$.

(25) Let $f(N) = rN\left(1-\frac{N}{K}\right) = rN - \frac{r}{K}N^2$.

Then $\underline{P_1(N) = rN}$ (about $a=0$) since $f(N)$ is a polynomial itself,

so $\underline{f(N) \approx rN}$ for $N \approx 0$.

(or use $\underline{f(0)=0}$ and $f'(N) = r - \frac{2r}{K}N$ so $\underline{f'(0) = r}$)

(26) a) $f(R) = \dfrac{aR}{K+R}$ (a, K constants)

$\underline{f'(R)} = \dfrac{(K+R)(a) - aR(1)}{(K+R)^2} = \dfrac{aK}{(K+R)^2}$, so $\underline{f'(0)} = \dfrac{aK}{K^2} = \dfrac{a}{K}$.

Then expanding about $R=0$ gives

$\underline{P_1(R)} = f(0) + f'(0)R = 0 + \frac{a}{K}R = \underline{\frac{a}{K}}R$, so $\underline{f(R) \approx \frac{a}{K}R}$ for $R \approx 0$.

b) Since $\underline{f(K)} = \dfrac{aK}{2K} = \dfrac{a}{2}$ and $\underline{f'(K)} = \dfrac{aK}{(2K)^2} = \dfrac{aK}{4K^2} = \dfrac{a}{4K}$,

expanding about $R=K$ gives

$\underline{P_1(R)} = f(K) + f'(K)(R-K) = \underline{\frac{a}{2} + \frac{a}{4K}(R-K)}$; so $\underline{f(R) \approx \frac{a}{2} + \frac{a}{4K}(R-K)}$ for $R \approx K$

(27) $f(x) = e^x$, $a=0$; $x=2$

Since $f^{(n)}(x) = e^x$ for all $n \geq 1$,

$$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| = \frac{e^c}{(n+1)!}(2-0)^{n+1} = \frac{e^c 2^{n+1}}{(n+1)!}$$ where c is between 0 and 2,

so $e^c < e^2 < 8 \Rightarrow |R_n(x)| < \dfrac{8 \cdot 2^{n+1}}{(n+1)!} < \dfrac{1}{1000}$ if $\dfrac{2^n}{(n+1)!} < \dfrac{1}{16{,}000}$ or

$\dfrac{(n+1)!}{2^n} > 16{,}000$, and this is satisfied for $\boxed{n=10}$.

(28) $f(x) = \cos x$, $a=0$; $x=1$

$f^{(n)}(x) = \pm\cos x$ if n is $\underline{\text{even}}$ and $f^{(n)}(x) = \pm\sin x$ if n is $\underline{\text{odd}}$,

so $$|R_n(x)| = \left|\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}\right| = \frac{|f^{(n+1)}(c)|}{(n+1)!}(1-0)^{n+1} = \frac{|f^{(n+1)}(c)|}{(n+1)!} \leq \frac{1}{(n+1)!}$$

(since $|f^{(n+1)}(c)| \leq 1$ for all n),

so $|R_n(x)| \leq \dfrac{1}{(n+1)!} < \dfrac{1}{100}$ if $(n+1)! > 100$, which is true for $\boxed{n=4}$