(23) $\frac{dL}{dT} = K(L_\infty - L)$, $L(0)=1$, $\boxed{L_\infty = 123}$, $L(27) = \frac{1}{2}(123)$

a) 1) $\frac{dL}{dT} + KL = KL_\infty$ so 2) $u(T) = e^{\int K\,dT} = e^{KT}$, 3) $e^{KT}\left[\frac{dL}{dT} + KL\right] = KL_\infty e^{KT}$,

$(e^{KT}L)' = L_\infty K e^{KT}$ 4) $e^{KT}L = L_\infty \int K e^{KT}\,dT = L_\infty e^{KT} + C$ so

$L(T) = L_\infty + Ce^{-KT} = 123 + Ce^{-KT}$ $L(0) = 123 + C = 1$, so $C = -122$

$L(27) = 123 - 122e^{-27K} = \frac{1}{2}(123)$ so $\frac{1}{2}(123) = 122e^{-27K}$, $e^{-27K} = \frac{123}{244}$,

$-27K = \ln\frac{123}{244}$, $K = \boxed{-\frac{1}{27}\ln\frac{123}{244}} = \boxed{\frac{1}{27}\ln\frac{244}{123}}$

b) $L(10) = 123 - 122e^{-10\left(-\frac{1}{27}\ln\frac{123}{244}\right)} = 123 - 122\left(e^{\ln\frac{123}{244}}\right)^{\frac{10}{27}} = \boxed{123 - 122\left(\frac{123}{244}\right)^{\frac{10}{27}}\text{ in}} \approx 28.34\text{ in}$

c) if $L(T) = .90(123)$, $123 - 122e^{-KT} = .90(123)$, $.1(123) = 122e^{-KT}$

$e^{-KT} = \frac{123}{1220}$, $-KT = \ln\left(\frac{123}{1220}\right)$, $T = -\frac{1}{K}\ln\left(\frac{123}{1220}\right) = \boxed{27\,\frac{\ln\left(\frac{123}{1220}\right)}{\ln\left(\frac{123}{244}\right)}\text{ mo}} \approx 90.4\text{ mo}$

(29) $\frac{dY}{dX} = 2Y(3-Y)$, $Y=5$ when $X=1$

$\int \frac{1}{Y(3-Y)}\,dY = \int 2\,dX$ $\qquad \frac{1}{Y(3-Y)} = \frac{A}{Y} + \frac{B}{3-Y}$

$1 = A(3-Y) + BY$

$Y=0$: $1 = 3A$ $\quad A = 1/3$

$Y=3$: $1 = 3B$ $\quad B = 1/3$

$\int\left(\frac{1/3}{Y} + \frac{1/3}{3-Y}\right)dY = 2X + C$

$\frac{1}{3}\int\left(\frac{1}{Y} - \frac{-1}{3-Y}\right)dY = 2X + C$ $\quad \frac{1}{3}\left(\ln Y - \ln(3-Y)\right) = 2X + C$ $\quad \ln\frac{Y}{3-Y} = 6X + D$

$\frac{Y}{3-Y} = e^{6X+D} = Ae^{6X}$, $\frac{3-Y}{Y} = \frac{1}{Ae^{6X}}$, $\frac{3}{Y} - 1 = ae^{-6X}$, $\frac{3}{Y} = 1 + ae^{-6X}$,

$\frac{Y}{3} = \frac{1}{1+ae^{-6X}}$, $Y = \frac{3}{1+ae^{-6X}}$ $\qquad$ when $X=1$, $Y=5$; so $5 = \frac{3}{1+ae^{-6}}$

$1 + ae^{-6} = \frac{3}{5}$, $ae^{-6} = -\frac{2}{5}$, $a = -\frac{2}{5}e^{6}$ $\qquad \boxed{Y = \frac{3}{1-\frac{2}{5}e^{6}(e^{-6X})} = \frac{3}{1-\frac{2}{5}e^{6-6X}}}$

(37) $\frac{dN}{dT} = .34N\left(1 - \frac{N}{200}\right) = .0017N(200-N)$, $N(0) = 50$

$\int\frac{1}{N(200-N)}\,dN = \int .0017\,dT$ $\qquad \frac{1}{N(200-N)} = \frac{A}{N} + \frac{B}{200-N}$

$1 = A(200-N) + BN$

$N=0$: $1 = 200A$ $\quad A = 1/200$

$N=200$: $1 = 200B$ $\quad B = 1/200$

$\int\left(\frac{\frac{1}{200}}{N} + \frac{\frac{1}{200}}{200-N}\right)dN = .0017T + C$

$\frac{1}{200}\int\left(\frac{1}{N} - \frac{-1}{200-N}\right)dN = .0017T + C$, $\quad \ln N - \ln(200-N) = .34T + D$, $\quad \ln\left(\frac{N}{200-N}\right) = .34T + D$

$\frac{N}{200-N} = e^{.34T+D} = Ae^{.34T}$, $\frac{200-N}{N} = \frac{1}{Ae^{.34T}}$, $\frac{200}{N} - 1 = ae^{-.34T}$, $\frac{200}{N} = 1 + ae^{-.34T}$,

$\frac{N}{200} = \frac{1}{1+ae^{-.34T}}$, $N = \frac{200}{1+ae^{-.34T}}$ $\qquad N(0) = \frac{200}{1+a\cdot 1} = 50$ so $1 + a = 4$, $a = 3$

$\boxed{N(T) = \frac{200}{1+3e^{-.34T}}}$ and $\lim_{T\to\infty} N(T) = \frac{200}{1+3(0)} = \boxed{200}$

REMARK NOTICE THAT THIS IS AN EXAMPLE OF LOGISTIC GROWTH.

(39a) $\frac{dN}{dt} = 1.5N\left(1-\frac{N}{50}\right) = .03N(50-N)$, $N(0)=10$

$\int \frac{1}{N(50-N)}\,dN = \int .03\,dt$ $\qquad \frac{1}{N(50-N)} = \frac{A}{N} + \frac{B}{50-N}$

$1 = A(50-N) + BN$

$\underline{N=0}$: $1 = 50A$ $\quad \underline{A = 1/50}$

$\underline{N=50}$: $1 = 50B$ $\quad \underline{B = 1/50}$

$\int \left(\frac{1/50}{N} + \frac{1/50}{50-N}\right) dN = .03t + C$

$\frac{1}{50}\int\left(\frac{1}{N} - \frac{-1}{50-N}\right) dN = .03t + C$, $\quad \frac{1}{50}\left(\ln N - \ln(50-N)\right) = .03t + C$, $\quad \ln\left(\frac{N}{50-N}\right) = 1.5t + D$

$\frac{N}{50-N} = e^{1.5t+D} = Ae^{1.5t}$, $\quad \frac{50-N}{N} = \frac{1}{Ae^{1.5t}}$, $\quad \frac{50}{N} - 1 = ae^{-1.5t}$, $\quad \frac{50}{N} = 1 + ae^{-1.5t}$

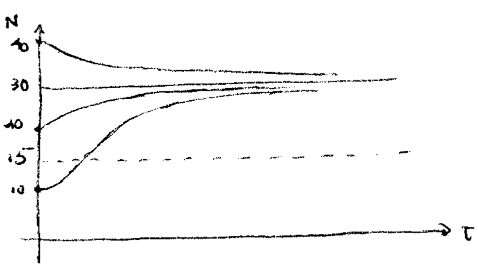
$\frac{N}{50} = \frac{1}{1+ae^{-1.5t}}$, $\quad \underline{N = \frac{50}{1+ae^{-1.5t}}}$ $\qquad N(0) = \frac{50}{1+a\cdot 1} = 10$ so $1+a=5$ and $\underline{a=4}$

$$\boxed{N(t) = \frac{50}{1+4e^{-1.5t}}}$$

REMARK This is a special case of logistic growth.

(41) a) $\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)$ gives $\boxed{\frac{dN}{dt} = 5N\left(1-\frac{N}{30}\right)}$

b)



(44) $\frac{dy}{dx} = 2\frac{y}{x}$, $y=1$ when $x=1$ $\qquad \int \frac{1}{y}\,dy = \int \frac{2}{x}\,dx$ $\qquad \ln y = 2\ln x + C$ $\qquad y = e^{2\ln x + C}$ so

$\underline{y} = Ae^{2\ln x} = A(e^{\ln x})^2 = \underline{Ax^2}$ $\qquad$ since $1 = A(1)^2$, $\underline{A=1}$ and $\boxed{y = x^2}$

REMARK Notice that this is also a first-order linear DE.

(47) $\frac{dy}{dx} = (y+1)e^{-x}$, $y=2$ when $x=0$

$\int \frac{1}{y+1}\,dy = \int e^{-x}\,dx$, $\quad \ln(y+1) = -e^{-x} + C$, $\quad y+1 = e^{(-e^{-x}+c)} = Ae^{(-e^{-x})}$,

$\underline{y = Ae^{-e^{-x}} - 1}$ $\qquad$ since $2 = A\cdot e^{-1} - 1$, $\quad Ae^{-1} = 3$ and $\underline{A = 3e}$

$$\boxed{y = (3e)e^{-e^{-x}} - 1 = 3e^{1-e^{-x}} - 1}$$

(48) $\frac{dy}{dx} = x^2y^2$, $y=1$ if $x=1$

$\int \frac{1}{y^2}\,dy = \int x^2\,dx$, $\quad -\frac{1}{y} = \frac{x^3}{3} + C$, $\quad \frac{1}{y} = D - \frac{x^3}{3}$, $\quad \underline{y} = \frac{1}{D-\frac{x^3}{3}} = \underline{\frac{3}{A-x^3}}$

since $1 = \frac{3}{A-1}$, $\quad A-1=3$, $\underline{A=4}$, and $\boxed{y = \frac{3}{4-x^3}}$

(53) $\log C = .8\log m + b$, so differentiating with respect to m gives $\frac{C'}{(\ln 10)C} = .8\left(\frac{1}{(\ln 10)m}\right)$

so $\boxed{\frac{dC}{dm} = \frac{.8C}{m}}$

MATH 17B — SOLUTIONS – GOLD SHEET

① $y' + y = e^{-x}$

2) $\underline{u(x)} = e^{\int 1\,dx} = \underline{e^x}$

3) $e^x\left[y' + y\right] = e^x\left[e^{-x}\right]$ so $(e^x y)' = 1$

4) $e^x y = \int 1\,dx = x + C$

$\boxed{y = xe^{-x} + Ce^{-x}}$

② $y' + \frac{y}{x} = 3x + 4$

2) $\underline{u(x)} = e^{\int \frac{1}{x}\,dx} = e^{\ln x} = \underline{x}$

3) $x\left[y' + \frac{y}{x}\right] = x\left[3x + 4\right]$ so $(xy)' = 3x^2 + 4x$

4) $xy = \int (3x^2 + 4x)\,dx = x^3 + 2x^2 + C$

$\boxed{y = x^2 + 2x + Cx^{-1}}$

③ $y' + \frac{2y}{x} = 4x + 3$

2) $\underline{u(x)} = e^{\int \frac{2}{x}\,dx} = e^{2\ln x} = (e^{\ln x})^2 = \underline{x^2}$

3) $x^2\left[y' + \frac{2}{x}y\right] = x^2\left[4x + 3\right]$ so $(x^2 y)' = 4x^3 + 3x^2$

4) $x^2 y = \int (4x^3 + 3x^2)\,dx = x^4 + x^3 + C$

$\boxed{y = x^2 + x + Cx^{-2}}$

④ $xy' - y = x^3 \ln x$

1) $y' - \frac{y}{x} = x^2 \ln x$

2) $\underline{u(x)} = e^{\int -\frac{1}{x}\,dx} = e^{-\ln x} = (e^{\ln x})^{-1} = \underline{x^{-1}}$

3) $x^{-1}\left[y' - \frac{y}{x}\right] = x^{-1}\left[x^2 \ln x\right]$ so $(x^{-1}y)' = x \ln x$

4) $x^{-1}y = \int x \ln x\,dx$ $\qquad \dfrac{u = \ln x, \quad dv = x\,dx}{du = \frac{1}{x}\,dx, \quad v = \frac{x^2}{2}}$

$x^{-1}y = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\left(\frac{x^2}{2}\right) + C$

$\boxed{y = \frac{x^3}{2}\ln x - \frac{x^3}{4} + Cx}$

⑤ $xy' - 2y = x^4 \sin x$

1) $y' - \frac{2}{x}y = x^3 \sin x$

2) $\underline{u(x)} = e^{\int -\frac{2}{x}\,dx} = e^{-2\ln x} = (e^{\ln x})^{-2} = \underline{x^{-2}}$

3) $x^{-2}\left[y' - \frac{2}{x}y\right] = x^{-2}\left[x^3 \sin x\right]$ so $(x^{-2}y)' = x \sin x$

4) $x^{-2}y = \int x \sin x\,dx$ $\qquad \dfrac{u = x, \quad dv = \sin x\,dx}{du = dx, \quad v = -\cos x}$

$x^{-2}y = -x\cos x - \int -\cos x\,dx = -x\cos x + \sin x + C$

$\boxed{y = -x^3 \cos x + x^2 \sin x + Cx^2}$