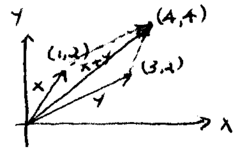
MATH 17B $\qquad$ Sec. 9.3 – SOLUTIONS

(17) $X = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $Y = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ so $X + Y = \underline{\begin{bmatrix} 4 \\ 4 \end{bmatrix}}$ :




$X+Y$ IS THE DIAGONAL OF THE PARALLELOGRAM DETERMINED BY $X$ AND $Y$

(37)
$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{2} & -\sin\frac{\pi}{2} \\ \sin\frac{\pi}{2} & \cos\frac{\pi}{2} \end{bmatrix},$$
so $A$ CORRESPONDS TO $\underline{\text{A ROTATION OF } \theta = \frac{\pi}{2}}$.

(40)
$$A = \begin{bmatrix} \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & -\sin\frac{\pi}{4} \\ \sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix},$$
so $A$ CORRESPONDS TO $\underline{\text{A ROTATION OF } \theta = \frac{\pi}{4}}$.

(41)
$$A\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{6} & -\sin\frac{\pi}{6} \\ \sin\frac{\pi}{6} & \cos\frac{\pi}{6} \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} \sqrt{3}/2 & -1/2 \\ 1/2 & \sqrt{3}/2 \end{bmatrix}\begin{bmatrix} -1 \\ 2 \end{bmatrix} = \boxed{\begin{bmatrix} -\frac{\sqrt{3}}{2} - 1 \\ -\frac{1}{2} + \sqrt{3} \end{bmatrix}}$$

(42)
$$A\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{3} & -\sin\frac{\pi}{3} \\ \sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}\begin{bmatrix} 4 \\ -1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 + \frac{\sqrt{3}}{2} \\ 2\sqrt{3} - \frac{1}{2} \end{bmatrix}}$$

(45)
$$A = \begin{bmatrix} \cos(-\frac{\pi}{4}) & -\sin(-\frac{\pi}{4}) \\ \sin(-\frac{\pi}{4}) & \cos(-\frac{\pi}{4}) \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{4} & \sin\frac{\pi}{4} \\ -\sin\frac{\pi}{4} & \cos\frac{\pi}{4} \end{bmatrix} = \begin{bmatrix} \sqrt{2}/2 & \sqrt{2}/2 \\ -\sqrt{2}/2 & \sqrt{2}/2 \end{bmatrix},$$
so $A\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{3\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} \end{bmatrix}}$

(46)
$$A = \begin{bmatrix} \cos(-\frac{\pi}{3}) & -\sin(-\frac{\pi}{3}) \\ \sin(-\frac{\pi}{3}) & \cos(-\frac{\pi}{3}) \end{bmatrix} = \begin{bmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{bmatrix} = \begin{bmatrix} 1/2 & \sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix},$$
so $A\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} \frac{1}{2} - \sqrt{3} \\ -\frac{\sqrt{3}}{2} - 1 \end{bmatrix}}$

(49)
$$A = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} \qquad \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 3 \\ 0 & -1-\lambda \end{vmatrix} = (2-\lambda)(-1-\lambda) - 0 = (\lambda - 2)(\lambda + 1) = 0$$
if $\boxed{\lambda = 2}$ or $\boxed{\lambda = -1}$

$\underline{\lambda = 2}$ : $\underline{(A - 2I)X = 0}$ gives
$$\begin{bmatrix} 0 & 3 & 0 \\ 0 & -3 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$
let $\underline{x = t}$, so $x = t$, $y = 0$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 0 \end{bmatrix} = t\begin{bmatrix} 1 \\ 0 \end{bmatrix},$$
so $\boxed{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ is an eigenvector corresp. to $\lambda = 2$

$\underline{\lambda = -1}$ : $\underline{(A - (-1)I)X = 0}$ gives
$$\begin{bmatrix} 3 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{cc} x & y \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$$
let $\underline{y = t}$, so $x = -t$, $y = t$

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix},$$
so $\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$ is an eigenvector corresp. to $\lambda = -1$

<u>REMARK</u> Since $A$ is an upper triangular matrix, its eigenvalues are the entries on the main diagonal.

(<u>CHECK</u> : $\underline{A\begin{bmatrix} 1 \\ 0 \end{bmatrix}} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \underline{2\begin{bmatrix} 1 \\ 0 \end{bmatrix}}$ and

$\underline{A\begin{bmatrix} -1 \\ 1 \end{bmatrix}} = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \underline{-1\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$ )

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## Sec. 9.3 – Solutions

(53) $A = \begin{bmatrix} -4 & 2 \\ -3 & 1 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} -4-\lambda & 2 \\ -3 & 1-\lambda \end{vmatrix} = (-4-\lambda)(1-\lambda) - (-6) = \lambda^2 + 3\lambda + 2 = (\lambda+2)(\lambda+1) = 0$$

IF $\boxed{\lambda = -2}$ OR $\boxed{\lambda = -1}$

$\underline{\lambda = -2}$: $\underline{(A-(-2)I)x = 0}$ GIVES $\begin{bmatrix} -2 & 2 & 0 \\ -3 & 3 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ LET $\underline{y = t}$, SO $x = t$, $y = t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, SO $\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ IS AN EIGENVECTOR CORRESP. TO $\lambda = -2$.

$\underline{\lambda = -1}$: $\underline{(A-(-1)I)x = 0}$ GIVES $\begin{bmatrix} -3 & 2 & 0 \\ -3 & 2 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & -2/3 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ LET $\underline{y = 3t}$, SO $x = 2t$, $y = 3t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2t \\ 3t \end{bmatrix} = t\begin{bmatrix} 2 \\ 3 \end{bmatrix}$, SO $\boxed{\begin{bmatrix} 2 \\ 3 \end{bmatrix}}$ IS AN EIGENVECTOR CORRESP. TO $\lambda = -1$.

(54) $A = \begin{bmatrix} 3 & 6 \\ -1 & -4 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 6 \\ -1 & -4-\lambda \end{vmatrix} = (3-\lambda)(-4-\lambda) - (-6) = \lambda^2 + \lambda - 6 = (\lambda+3)(\lambda-2) = 0$$

IF $\boxed{\lambda = -3}$ OR $\boxed{\lambda = 2}$

$\underline{\lambda = -3}$: $\underline{(A-(-3)I)x = 0}$ GIVES $\begin{bmatrix} 6 & 6 & 0 \\ -1 & -1 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ LET $\underline{y = t}$, SO $x = -t$, $y = t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, SO $\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$ IS AN EIGENVECTOR CORRESP. TO $\lambda = -3$

$\underline{\lambda = 2}$: $\underline{(A-2I)x = 0}$ GIVES $\begin{bmatrix} 1 & 6 & 0 \\ -1 & -6 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ LET $\underline{y = t}$, SO $x = -6t$, $y = t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -6t \\ t \end{bmatrix} = t\begin{bmatrix} -6 \\ 1 \end{bmatrix}$, SO $\boxed{\begin{bmatrix} -6 \\ 1 \end{bmatrix}}$ IS AN EIGENVECTOR CORRESP. TO $\lambda = 2$

(55) $A = \begin{bmatrix} 2 & 1 \\ 2 & 3 \end{bmatrix}$

$$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 \\ 2 & 3-\lambda \end{vmatrix} = (2-\lambda)(3-\lambda) - 2 = \lambda^2 - 5\lambda + 4 = (\lambda-4)(\lambda-1) = 0$$

IF $\boxed{\lambda = 4}$ OR $\boxed{\lambda = 1}$

$\underline{\lambda = 4}$: $\underline{(A-4I)x = 0}$ GIVES $\begin{bmatrix} -2 & 1 & 0 \\ 2 & -1 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ LET $\underline{y = 2t}$, SO $x = t$, $y = 2t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ 2t \end{bmatrix} = t\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, SO $\boxed{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}$ IS AN EIGENVECTOR CORRESP. TO $\lambda = 4$

$\underline{\lambda = 1}$: $\underline{(A-I)x = 0}$ GIVES $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \end{bmatrix} \longrightarrow \begin{array}{c} \begin{array}{ccc} x & y & \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array}$ LET $\underline{y = t}$, SO $x = -t$, $y = t$

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -t \\ t \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, SO $\boxed{\begin{bmatrix} -1 \\ 1 \end{bmatrix}}$ IS AN EIGENVECTOR CORRESP. TO $\lambda = 1$

<u>REMARK</u> NOTICE THAT ANOTHER WAY TO FIND THE EIGENVALUES FOR A $2 \times 2$ MATRIX IS TO USE THE EQUATION $\underline{\lambda^2 - \operatorname{tr}(A)\lambda + \det(A) = 0}$.