MATH 17B　　　SEC. 7.3 – SOLUTIONS

(13) $\int \frac{1}{x(x-2)}\,dx \qquad \frac{1}{x(x-2)} = \frac{A}{x} + \frac{B}{x-2} \qquad 1 = A(x-2) + Bx$

$\underline{x=0}: \; 1 = -2A \qquad \underline{A = -1/2}$

$\underline{x=2}: \; 1 = 2B \qquad \underline{B = 1/2}$

$$= \int \left(\frac{-1/2}{x} + \frac{1/2}{x-2}\right) dx = \frac{1}{2}\int\left(\frac{1}{x-2} - \frac{1}{x}\right)dx = \boxed{\frac{1}{2}\left[\ln|x-2| - \ln|x|\right] + C} = \boxed{\frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C}$$

(15) $\int \frac{1}{(x+1)(x-3)}\,dx \qquad \frac{1}{(x+1)(x-3)} = \frac{A}{x+1} + \frac{B}{x-3}$

$1 = A(x-3) + B(x+1)$

$\underline{x=-1}: \; 1 = -4A \qquad \underline{A = -1/4}$

$\underline{x=3}: \; 1 = 4B \qquad \underline{B = 1/4}$

$$= \int\left(\frac{-1/4}{x+1} + \frac{1/4}{x-3}\right)dx = \frac{1}{4}\int\left(\frac{1}{x-3} - \frac{1}{x+1}\right)dx = \boxed{\frac{1}{4}\left[\ln|x-3| - \ln|x+1|\right] + C} = \boxed{\frac{1}{4}\ln\left|\frac{x-3}{x+1}\right| + C}$$

(17) $\int \frac{x^2-2x-2}{x^2(x+2)}\,dx \qquad \frac{x^2-2x-2}{x^2(x+2)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+2}$

$x^2 - 2x - 2 = Ax(x+2) + B(x+2) + Cx^2$

$\underline{x=0}: \; -2 = 2B \qquad \underline{B=-1}$

$\underline{x=-2}: \; 6 = 4C \qquad \underline{C = 3/2}$

$\underline{x^2 \text{ COEFF.}}: \; 1 = A + C \qquad A = 1 - C \quad \text{so} \quad \underline{A = -1/2}$

$$= \int\left(\frac{-1/2}{x} + \frac{-1}{x^2} + \frac{3/2}{x+2}\right)dx = \boxed{-\frac{1}{2}\ln|x| + \frac{1}{x} + \frac{3}{2}\ln|x+2| + C}$$

(21) $\int \frac{2x^2-3x+2}{(x^2+1)^2}\,dx \qquad \frac{2x^2-3x+2}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$

$2x^2 - 3x + 2 = (Ax+B)(x^2+1) + (Cx+D)$

$= Ax^3 + Bx^2 + (A+C)x + (B+D)$

$\underline{x^3 \text{ COEFF.}}: \; \underline{0 = A}$

$\underline{x^2 \text{ COEFF.}}: \; \underline{2 = B}$

$\underline{x \text{ COEFF.}}: \; -3 = A + C \quad \text{so} \quad \underline{C = -3}$

$\underline{x = 0}: \; 2 = B + D \quad \text{so} \quad \underline{D = 0}$

$$= \int\left(\frac{2}{x^2+1} + \frac{-3x}{(x^2+1)^2}\right)dx = 2\tan^{-1}x - \frac{3}{2}\int \frac{1}{(x^2+1)^2}\cdot 2x\,dx \qquad \text{LET } \underline{u = x^2+1}, \; du = 2x\,dx$$

$$= 2\tan^{-1}x - \frac{3}{2}\int \frac{1}{u^2}\,du = 2\tan^{-1}x - \frac{3}{2}\left(-\frac{1}{u}\right) + C = \boxed{2\tan^{-1}x + \frac{3}{2}\cdot\frac{1}{x^2+1} + C}$$

(23) $\int \frac{1}{x^2-2x+2}\,dx = \int \frac{1}{(x-1)^2+1}\,dx \qquad \text{LET } \underline{u = x-1}, \; du = dx$

$$= \int \frac{1}{u^2+1}\,du = \tan^{-1}u + C = \boxed{\tan^{-1}(x-1) + C}$$

(25) $\int \frac{1}{x^2-4x+13}\,dx = \int \frac{1}{(x-2)^2+9}\,dx \qquad \text{LET } \underline{u = x-2}, \; du = dx$

$$= \int \frac{1}{u^2+9}\,du = \frac{1}{3}\tan^{-1}\frac{u}{3} + C = \boxed{\frac{1}{3}\tan^{-1}\left(\frac{x-2}{3}\right) + C}$$

(USING $x^2 - 4x + 13 = (x^2 - 4x + \underline{4}) + \underline{9} = (x-2)^2 + 9$)

(29) $\displaystyle\int \frac{1}{x^2-9}\,dx \qquad \frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)} = \frac{A}{x-3} + \frac{B}{x+3}$

$1 = A(x+3) + B(x-3)$

$x=3$: $1 = 6A$ $\quad A = 1/6$

$x=-3$: $1 = -6B$ $\quad B = -1/6$

$$= \int\left(\frac{1/6}{x-3} - \frac{1/6}{x+3}\right)dx = \frac{1}{6}\int\left(\frac{1}{x-3} - \frac{1}{x+3}\right)dx = \boxed{\frac{1}{6}\left[\ln|x-3| - \ln|x+3|\right] + C} = \boxed{\frac{1}{6}\ln\left|\frac{x-3}{x+3}\right| + C}$$

(31) $\displaystyle\int \frac{1}{x^2-x-2}\,dx \qquad \frac{1}{x^2-x-2} = \frac{1}{(x-2)(x+1)} = \frac{A}{x-2} + \frac{B}{x+1}$

$1 = A(x+1) + B(x-2)$

$x=2$: $1 = 3A$ $\quad A = 1/3$

$x=-1$: $1 = -3B$ $\quad B = -1/3$

$$= \int\left(\frac{1/3}{x-2} - \frac{1/3}{x+1}\right)dx = \frac{1}{3}\int\left(\frac{1}{x-2} - \frac{1}{x+1}\right)dx = \boxed{\frac{1}{3}\left[\ln|x-2| - \ln|x+1|\right] + C} = \boxed{\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + C}$$

(33) $\displaystyle\int \frac{x^2+1}{x^2+3x+2}\,dx = \int\left(1 + \frac{-3x-1}{x^2+3x+2}\right)dx \longleftarrow$

$$\begin{array}{r} 1 \\ x^2+3x+2 \,\overline{)\, x^2 \qquad +1} \\ x^2+3x+2 \\ \hline -3x-1 \end{array}$$

$$= x - \int \frac{3x+1}{(x+1)(x+2)}\,dx \qquad \frac{3x+1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$3x+1 = A(x+2) + B(x+1)$

$x=-1$: $-2 = A$

$x=-2$: $-5 = -B$ so $B=5$

$$= x - \int\left(\frac{-2}{x+1} + \frac{5}{x+2}\right)dx$$

$$= \boxed{x + 2\ln|x+1| - 5\ln|x+2| + C}$$

(34) $\displaystyle\int \frac{x^3+1}{x^2+3}\,dx = \int\left(x + \frac{-3x+1}{x^2+3}\right)dx \longleftarrow$

$$\begin{array}{r} x \\ x^2+3 \,\overline{)\, x^3 \qquad +1} \\ x^3+3x \\ \hline -3x+1 \end{array}$$

$$= \frac{x^2}{2} + \int \frac{-3x+1}{x^2+3}\,dx = \frac{x^2}{2} - \frac{3}{2}\int\frac{2x}{x^2+3}\,dx + \int\frac{1}{x^2+3}\,dx$$

$$= \boxed{\frac{x^2}{2} - \frac{3}{2}\ln(x^2+3) + \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C}$$

(since $x^2+3>0$ for all x, we can use $\ln(x^2+3)$ instead of $\ln|x^2+3|$)

(38) $\displaystyle\int_3^5 \frac{x}{x-1}\,dx = \int_3^5\left(1 + \frac{1}{x-1}\right)dx \longleftarrow \frac{x}{x-1} = \frac{(x-1)+1}{x-1} = \frac{x-1}{x-1} + \frac{1}{x-1} = 1 + \frac{1}{x-1}$

$$= \Big[x + \ln|x-1|\Big]_3^5 = (5+\ln 4) - (3+\ln 2) = \boxed{2+\ln 2}$$

REMARK We could have used $\ln(x-1)$ here, since $x-1>0$ for $3 \le x \le 5$; and we could have also used the substitution $u = x-1$, $x = u+1$, $dx = du$.

(43) $\displaystyle\int_0^1 \tan^{-1}x\,dx$ Let $u = \tan^{-1}x$, $dv = dx$

$du = \frac{1}{x^2+1}\,dx$, $v = x$

$$= \left[x\tan^{-1}x - \int\frac{x}{x^2+1}\,dx\right]_0^1 = \left[x\tan^{-1}x - \frac{1}{2}\int\frac{2x}{x^2+1}\,dx\right]_0^1$$

$$= \left[x\tan^{-1}x - \frac{1}{2}\ln(x^2+1)\right]_0^1 = \left(1\cdot\frac{\pi}{4} - \frac{1}{2}\ln 2\right) - (0-0) = \boxed{\frac{\pi}{4} - \frac{1}{2}\ln 2} = \boxed{\frac{\pi - 2\ln 2}{4}}$$