Sec. 7.3 – SOLUTIONS

(22) $\displaystyle\int \frac{3x^2+4x+3}{(x^2+1)^2}\,dx$

$$\frac{3x^2+4x+3}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

$$3x^2+4x+3 = (Ax+B)(x^2+1) + (Cx+D)$$
$$= Ax^3 + Bx^2 + (A+C)x + (B+D)$$

x^3 coeff. : $\underline{0 = A}$

x^2 coeff. : $\underline{3 = B}$

x coeff. : $4 = A + C$ so $\underline{C = 4}$

$x = 0$: $3 = B + D$ so $\underline{D = 0}$

$$\int \left(\frac{3}{x^2+1} + \frac{4x}{(x^2+1)^2}\right) dx = 3\tan^{-1}x + 2\int \frac{1}{(x^2+1)^2}\cdot 2x\,dx \qquad \underline{u = x^2+1},\ du = 2x\,dx$$

$$= 3\tan^{-1}x + 2\int \frac{1}{u^2}\,du = 3\tan^{-1}x + 2(-u^{-1}) + C = \boxed{3\tan^{-1}x - 2(x^2+1)^{-1} + C}$$

<u>REMARK</u> We could also use $\dfrac{3x^2+4x+3}{(x^2+1)^2} = \dfrac{3(x^2+1)+4x}{(x^2+1)^2} = \dfrac{3}{x^2+1} + \dfrac{4x}{(x^2+1)^2}$ to get the partial fraction expansion.

(36) $$\int \frac{x^4+3}{x^2-4x+3}\,dx = \int \left(x^2+4x+13 + \frac{40x-36}{x^2-4x+3}\right) dx$$

$$\begin{array}{r} x^2+4x+13 \\ x^2-4x+3\ \overline{)\ x^4 \qquad\qquad\quad +3} \\ \underline{x^4-4x^3+3x^2} \\ 4x^3-3x^2 \\ \underline{4x^3-16x^2+12x} \\ 13x^2-12x+3 \\ \underline{13x^2-52x+39} \\ 40x-36 \end{array}$$

$$= \frac{x^3}{3} + 2x^2 + 13x + \int \frac{40x-36}{x^2-4x+3}\,dx$$

$$\frac{40x-36}{(x-1)(x-3)} = \frac{A}{x-1} + \frac{B}{x-3}$$

$$40x - 36 = A(x-3) + B(x-1)$$

$\underline{x=1}$: $4 = -2A$ $\quad \underline{A = -2}$

$\underline{x=3}$: $84 = 2B$ $\quad \underline{B = 42}$

$$= \frac{x^3}{3} + 2x^2 + 13x + \int\left(\frac{-2}{x-1} + \frac{42}{x-3}\right) dx$$

$$= \boxed{\frac{x^3}{3} + 2x^2 + 13x - 2\ln|x-1| + 42\ln|x-3| + C}$$

(42) $\displaystyle\int_2^3 \frac{1}{1-x^2}\,dx$

$$\frac{1}{1-x^2} = \frac{1}{(1-x)(1+x)} = \frac{A}{1-x} + \frac{B}{1+x}$$

$$1 = A(1+x) + B(1-x)$$

$\underline{x=1}$: $1 = 2A$ $\quad \underline{A = 1/2}$

$\underline{x=-1}$: $1 = 2B$ $\quad \underline{B = 1/2}$

$$= \int_2^3 \left(\frac{1/2}{1-x} + \frac{1/2}{1+x}\right) dx = \frac{1}{2}\int_2^3 \left(\frac{1}{1+x} - \frac{1}{x-1}\right) dx$$

$$= \frac{1}{2}\Big[\ln|x+1| - \ln|x-1|\Big]_2^3 = \frac{1}{2}\left[\ln\left(\frac{x+1}{x-1}\right)\right]_2^3 = \frac{1}{2}(\ln 2 - \ln 3) = \boxed{\frac{1}{2}\ln\frac{2}{3}} = \boxed{\frac{1}{2}\ln\frac{2}{3}}$$

(44) $\displaystyle\int_0^1 x\tan^{-1}x\,dx$ $\qquad$ Let $\underline{u = \tan^{-1}x,\ dv = x\,dx}$

$$du = \frac{1}{x^2+1}\,dx,\quad v = \frac{1}{2}(x^2+1)$$

$$= \left[\frac{1}{2}(x^2+1)\tan^{-1}x - \int \frac{1}{2}\,dx\right]_0^1 = \left[\frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x\right]_0^1 = \left(\tan^{-1}1 - \frac{1}{2}\right) - \left(\frac{1}{2}\tan^{-1}0 - 0\right)$$

$$= \boxed{\frac{\pi}{4} - \frac{1}{2}} = \boxed{\frac{\pi-2}{4}}$$

$\boxed{\text{OR}}$ use $v = \frac{1}{2}x^2$ to get

$$\left[\frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{x^2}{x^2+1}\,dx\right]_0^1 = \left[\frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int \frac{(x^2+1)-1}{x^2+1}\,dx\right]_0^1$$

$$= \left[\frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}\int\left(1 - \frac{1}{x^2+1}\right)dx\right]_0^1 = \left[\frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x)\right]_0^1$$

$$= \left(\frac{1}{2}\tan^{-1}1 - \frac{1}{2} + \frac{1}{2}\tan^{-1}1\right) - 0 = \tan^{-1}1 - \frac{1}{2} = \boxed{\frac{\pi}{4} - \frac{1}{2}} = \boxed{\frac{\pi-2}{4}}$$

MATH 17B   Sec. 7.3 – SOLUTIONS

(45) $\displaystyle\int \frac{1}{x(x+1)^2}\,dx \qquad \frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$1 = A(x+1)^2 + Bx(x+1) + Cx$

$x=0$: $1 = A$

$x=-1$: $1 = -C$ so $C=-1$

x^2 coeff.: $0 = A+B$ so $B=-1$

$$\int\left(\frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}\right)dx = \ln|x| - \ln|x+1| - \int \frac{1}{u^2}\,du \quad (u = x+1,\ du = dx)$$

$$= \ln|x| - \ln|x+1| - (-u^{-1}) + C = \boxed{\ln|x| - \ln|x+1| + (x+1)^{-1} + C} = \boxed{\ln\left|\frac{x}{x+1}\right| + \frac{1}{x+1} + C}$$

(47) $\displaystyle\int \frac{4}{(1-x)(1+x)^2}\,dx \qquad \frac{4}{(1-x)(1+x)^2} = \frac{A}{1-x} + \frac{B}{1+x} + \frac{C}{(1+x)^2}$

$4 = A(1+x)^2 + B(1-x)(1+x) + C(1-x)$

$x=1$: $4 = 4A$ $\quad A=1$

$x=-1$: $4 = 2C$ $\quad C=2$

x^2 coeff.: $0 = A-B$ $\quad B=1$

$$\int\left(\frac{1}{1-x} + \frac{1}{1+x} + \frac{2}{(1+x)^2}\right)dx = -\int\frac{-1}{1-x}\,dx + \int\frac{1}{1+x}\,dx + 2\int\frac{1}{u^2}\,du \quad (u = 1+x,\ du = dx)$$

$$= -\ln|1-x| + \ln|1+x| + 2(-u^{-1}) + C = \boxed{\ln|1+x| - \ln|1-x| - 2(1+x)^{-1} + C}$$

$$= \boxed{\ln\left|\frac{1+x}{1-x}\right| - \frac{2}{1+x} + C}$$

(49) $\displaystyle\int \frac{1}{(x^2-9)^2}\,dx = \int \frac{1}{((x-3)(x+3))^2}\,dx$

$\displaystyle = \int \frac{1}{(x-3)^2(x+3)^2}\,dx \qquad \frac{1}{(x-3)^2(x+3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{C}{x+3} + \frac{D}{(x+3)^2}$

$1 = A(x-3)(x+3)^2 + B(x+3)^2 + C(x-3)^2(x+3) + D(x-3)^2$

$x=3$: $1 = 36B$ $\quad B = 1/36$

$x=-3$: $1 = 36D$ $\quad D = 1/36$

x^3 coeff.: $0 = A + C$ so $C=-A$

$x=0$: $1 = -27A + 9B + 27C + 9D = -27A + 27C + \frac{1}{4} + \frac{1}{4}$ so $-27A + 27C = \frac{1}{2}$,

$-27A - 27A = 1/2$, $-54A = 1/2$, $A = -\frac{1}{108}$, $C = \frac{1}{108}$

$$\int\left(\frac{-\frac{1}{108}}{x-3} + \frac{\frac{1}{36}}{(x-3)^2} + \frac{\frac{1}{108}}{x+3} + \frac{\frac{1}{36}}{(x+3)^2}\right)dx$$

$$= \frac{1}{108}\int\left(-\frac{1}{x-3} + \frac{3}{(x-3)^2} + \frac{1}{x+3} + \frac{3}{(x+3)^2}\right)dx = \boxed{\frac{1}{108}\left[-\ln|x-3| - 3(x-3)^{-1} + \ln|x+3| - 3(x+3)^{-1}\right] + C}$$

$$= \boxed{\frac{1}{108}\left[\ln\left|\frac{x+3}{x-3}\right| - \frac{3}{x-3} - \frac{3}{x+3}\right] + C}$$

(51) $\displaystyle\int \frac{1}{x^2(x^2+1)}\,dx \qquad \frac{1}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$

$1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$

$x=0$: $1 = B$

x coeff.: $0 = A$

x^3 coeff.: $0 = A + C$ so $C=0$

x^2 coeff.: $0 = B + D$ so $D=-1$

$$\int\left(\frac{1}{x^2} + \frac{-1}{x^2+1}\right)dx = \boxed{-\frac{1}{x} - \tan^{-1}x + C}$$

REMARK We can also use $\displaystyle\frac{1}{x^2(x^2+1)} = \frac{(x^2+1) - x^2}{x^2(x^2+1)} = \frac{\cancel{x^2+1}}{x^2\cancel{(x^2+1)}} - \frac{\cancel{x^2}}{\cancel{x^2}(x^2+1)} = \frac{1}{x^2} - \frac{1}{x^2+1}$

to get the partial fraction expansion.