MATH 17B

SOLUTIONS - TEST 2

① $\displaystyle\int \frac{x^2+20}{x(x-2)^2}\,dx$ $\qquad$ $\displaystyle\frac{x^2+20}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$

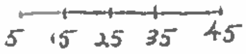
$x^2+20 = A(x-2)^2 + Bx(x-2) + Cx$

$\underline{x=2}$: $\quad 24 = 2C \quad \underline{C=12}$

$\underline{x=0}$: $\quad 20 = 4A \quad \underline{A=5}$

$\underline{x^2 \text{ COEFF.}}$: $\quad 1 = A + B = 5 + B \quad \underline{B=-4}$

$$\int\left(\frac{5}{x} - \frac{4}{x-2} + \frac{12}{(x-2)^2}\right)dx = \boxed{5\ln|x| - 4\ln|x-2| - 12(x-2)^{-1} + C}$$

② 

$\underline{\Delta x} = \dfrac{45-5}{4} = \dfrac{40}{4} = \underline{10}$

$$\int_5^{45} \frac{1}{x+6}\,dx \approx \boxed{\frac{10}{3}\left[\frac{1}{11} + \frac{4}{21} + \frac{2}{31} + \frac{4}{41} + \frac{1}{51}\right]}$$

③ $\displaystyle\int x^2 \sin 3x\,dx = \boxed{-\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + C}$ ←

$\underline{u}$		$\underline{dv}$
x^2	⊕	$\sin 3x\,dx$
$2x$	⊖	$-\frac{1}{3}\cos 3x$
2	⊕	$-\frac{1}{9}\sin 3x$
0		$\frac{1}{27}\cos 3x$

OR $\quad$ LET $u = x^2$, $\quad dv = \sin 3x\,dx$

$du = 2x\,dx$, $\quad v = -\frac{1}{3}\cos 3x$

$$\int x^2\sin 3x\,dx = -\frac{1}{3}x^2\cos 3x - \left(-\frac{2}{3}\right)\int x\cos 3x\,dx$$

LET $\underline{u = x, \quad dv = \cos 3x\,dx}$

$du = dx, \quad v = \frac{1}{3}\sin 5x$

$$= -\frac{1}{3}x^2\cos 3x + \frac{2}{3}\left[\frac{1}{3}x\sin 3x - \frac{1}{3}\int \sin 3x\,dx\right]$$

$$= \boxed{-\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x - \frac{2}{9}\left(-\frac{1}{3}\cos 3x\right) + C}$$

$$= \boxed{-\frac{1}{3}x^2\cos 3x + \frac{2}{9}x\sin 3x + \frac{2}{27}\cos 3x + C}$$

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④ $\displaystyle\int_1^3 \frac{\sqrt{x}}{x+1}\,dx$ $\qquad$ LET $\underline{u = \sqrt{x}}$, $\quad$ SO $x = u^2$, $\quad dx = 2u\,du$ $\qquad$ IF $x=1$, $u=1$; $\quad x=3$, $u=\sqrt{3}$

$$= \int_1^{\sqrt{3}} \frac{u}{u^2+1}\cdot 2u\,du = 2\int_1^{\sqrt{3}} \frac{u^2}{u^2+1}\,du = 2\int_1^{\sqrt{3}}\left(1 - \frac{1}{u^2+1}\right)du$$ ←

$$\begin{array}{r} 1 \\ u^2+1\,\overline{)\,u^2} \\ \underline{u^2+1} \\ -1 \end{array}$$

$$= 2\left[u - \tan^{-1}u\right]_1^{\sqrt{3}} = 2\left(\sqrt{3} - \tan^{-1}\sqrt{3} - (1 - \tan^{-1}1)\right) = 2\left(\sqrt{3} - \frac{\pi}{3} - 1 + \frac{\pi}{4}\right) = \boxed{2\sqrt{3} - 2 - \frac{\pi}{6}}$$

⑤ $\displaystyle\int_1^e \frac{\ln x}{x^2}\,dx$ $\qquad$ LET $\underline{u = \ln x, \quad dv = x^{-2}\,dx}$

$du = \frac{1}{x}\,dx, \quad v = -\frac{1}{x}$

$$= \left[(\ln x)\left(-\frac{1}{x}\right) - \int -\frac{1}{x^2}\,dx\right]_1^e = \left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^e = \left(-\frac{1}{e} - \frac{1}{e}\right) - (0 - 1) = \boxed{1 - \frac{2}{e}}$$

(SINCE $\ln e = 1$ AND $\ln 1 = 0$)

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⑥ $$\int_{3/8}^{\infty} \frac{12}{2x^2+3x}\,dx = \lim_{t\to\infty}\int_{3/8}^{t} \frac{12}{2x^2+3x}\,dx$$

$$\frac{12}{2x^2+3x} = \frac{12}{x(2x+3)} = \frac{A}{x} + \frac{B}{2x+3}$$

$$12 = A(2x+3) + Bx$$

$\underline{x=0:}$ $12 = 3A$ $\underline{A=4}$

$\underline{x \text{ coeff.}:}$ $0 = 2A + B$ $\underline{B=-8}$

$$= \lim_{t\to\infty}\int_{3/8}^{t}\left(\frac{4}{x} - \frac{8}{2x+3}\right)dx$$

$$= \lim_{t\to\infty} 4\int_{3/8}^{t}\left(\frac{1}{x} - \frac{2}{2x+3}\right)dx = \lim_{t\to\infty} 4\Big[\ln x - \ln(2x+3)\Big]_{3/8}^{t}$$

$$= \lim_{t\to\infty} 4\left[\ln\left(\frac{x}{2x+3}\right)\right]_{3/8}^{t} = \lim_{t\to\infty} 4\left(\ln\frac{t}{2t+3} - \ln\frac{3/8}{30/8}\right) = 4\left(\ln\frac{1}{2} - \ln\frac{1}{10}\right)$$

$$= 4\ln\frac{1/2}{1/10} = \boxed{4\ln 5}$$

(as $t\to\infty$, $\frac{t}{2t+3} \to \frac{1}{2}$ so $\ln\frac{t}{2t+3} \to \ln\frac{1}{2}$)

⑦ $$\int t\sec^2 t\,dt$$

Let $\underline{u = t,\quad dv = \sec^2 t\,dt}$

$du = dt,\quad v = \tan t$

$$= t\tan t - \int \tan t\,dt = t\tan t - \int\frac{\sin t}{\cos t}\,dt \leftarrow (u=\cos t,\ u' = -\sin t)$$

$$= t\tan t + \int\frac{-\sin t}{\cos t}\,dt = \boxed{t\tan t + \ln|\cos t| + C} = \boxed{t\tan t - \ln|\sec t| + C}$$

⑧ $f(x) = \ln x,\quad a = 1$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$

$f^{(4)}(x) = -6x^{-4}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$	
0	0	0	
1	1	1	
2	−1	−1/2	
3	2	1/3	$\leftarrow \frac{2}{6}$
4	−6	−1/4	$\leftarrow -\frac{6}{24}$

$$\boxed{P_4(x) = (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 - \tfrac{1}{4}(x-1)^4}$$

⑨ $$\int\frac{22x-21}{(x+2)(x^2+9)}\,dx$$

$$\frac{22x-21}{(x+2)(x^2+9)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+9}$$

$$22x - 21 = A(x^2+9) + (Bx+C)(x+2)$$

$\underline{x=-2:}$ $-65 = 13A$ $\underline{A=-5}$

$\underline{x=0:}$ $-21 = 9A + 2C = -45 + 2C$ $2C = 24$ $\underline{C=12}$

$\underline{x^2 \text{ coeff.}:}$ $0 = A + B$ $\underline{B=5}$

$$\int\left(\frac{-5}{x+2} + \frac{5x+12}{x^2+9}\right)dx = -5\int\frac{1}{x+2}\,dx + \int\frac{5x}{x^2+9}\,dx + \int\frac{12}{x^2+9}\,dx$$

$$= -5\ln|x+2| + \frac{5}{\textcircled{2}}\int\frac{\textcircled{2}x}{x^2+9}\,dx + 12\int\frac{1}{x^2+9}\,dx$$

$$= \boxed{-5\ln|x+2| + \frac{5}{2}\ln|x^2+9| + 12\left(\frac{1}{3}\tan^{-1}\frac{x}{3}\right) + C}$$

$$= \boxed{-5\ln|x+2| + \frac{5}{2}\ln(x^2+9) + 4\tan^{-1}\frac{x}{3} + C}$$

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⑩ $\frac{dy}{dx} = 10xy^2$, $y = 6$ WHEN $x = 2$

$\int \frac{1}{y^2}\,dy = \int 10x\,dx$ $\quad -\frac{1}{y} = 5x^2 + C$ $\quad \frac{1}{y} = -5x^2 + D$ $\quad \underline{y = \frac{1}{D - 5x^2}}$

SINCE $y = 6$ WHEN $x = 2$, $\frac{1}{6} = -5(4) + D$ SO $\underline{D = \frac{121}{6}}$ AND

$\boxed{y = \frac{1}{\frac{121}{6} - 5x^2}}$ OR $\boxed{y = \frac{6}{121 - 30x^2}}$

⑪ $xy' + 2y = 15x + 8$

1) $y' + \frac{2}{x}y = 15 + \frac{8}{x}$

2) $\underline{u(x)} = e^{\int \frac{2}{x}dx} = e^{2\ln x} = (e^{\ln x})^2 = \underline{x^2}$

3) $x^2\left[y' + \frac{2}{x}y\right] = x^2\left[15 + \frac{8}{x}\right]$

$(x^2 y)' = 15x^2 + 8x$

4) $\underline{x^2 y} = \int (15x^2 + 8x)\,dx = 5x^3 + 4x^2 + C$

$\boxed{y = 5x + 4 + \frac{C}{x^2}}$

⑫ $\int \frac{60}{3e^{2x} + 5}\,dx = \int \frac{60}{3e^{2x} + 5} \cdot \frac{e^{-2x}}{e^{-2x}}\,dx = \int \frac{60e^{-2x}}{3 + 5e^{-2x}}\,dx$ $\quad \frac{u = 3 + 5e^{-2x}}{u' = -10e^{-2x}}$

$= \frac{60}{\enclose{circle}{-10}} \int \frac{\enclose{circle}{-10}\, e^{-2x}}{3 + 5e^{-2x}}\,dx = \boxed{-6\ln|3 + 5e^{-2x}| + C} = \boxed{-6\ln(3 + 5e^{-2x}) + C}$

(SINCE $3 + 5e^{-2x} > 0$ FOR ALL x)

$\boxed{\text{OR}}$ LET $\underline{u = e^x}$ SO $x = \ln u$, $dx = \frac{1}{u}\,du$ TO GET

$\int \frac{60}{3u^2 + 5} \cdot \frac{1}{u}\,du = \int \frac{60}{u(3u^2 + 5)}\,du = \int \left(\frac{12}{u} - \frac{36u}{3u^2 + 5} \right) du \longleftarrow$ (USING PARTIAL FRACTIONS)

$= 12\ln|u| - 6\ln|3u^2 + 5| + C = 12\ln e^x - 6\ln(3e^{2x} + 5) + C$

$= \boxed{12x - 6\ln(3e^{2x} + 5) + C}$

$\boxed{\text{OR}}$ LET $\underline{u = e^{2x}}$ SO $x = \frac{1}{2}\ln u$, $dx = \frac{1}{2} \cdot \frac{1}{u}\,du$ TO GET

$\int \frac{60}{3u + 5} \cdot \frac{1}{2u}\,du = \int \frac{30}{u(3u + 5)}\,du = \int \left(\frac{6}{u} - \frac{18}{3u + 5} \right) du \longleftarrow$ (USING PARTIAL FRACTIONS)

$= 6\ln|u| - 6\ln|3u + 5| + C = 6\ln e^{2x} - 6\ln(3e^{2x} + 5) + C$

$= 6(2x) - 6\ln(3e^{2x} + 5) + C = \boxed{12x - 6\ln(3e^{2x} + 5) + C}$