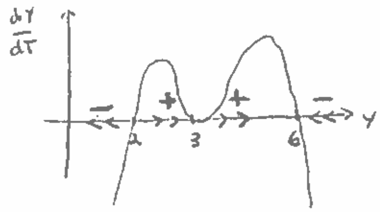
MATH 17B ANSWERS – TEST 3

① $\dfrac{dY}{dT} = 8(Y-2)(Y-3)^2(6-Y)$ $\qquad g(7) = 8 \cdot 5 \cdot 4^2(-1) < 0$




$$\boxed{\begin{array}{l} Y = 2 : \text{ UNSTABLE} \\ Y = 3 : \text{ SEMISTABLE} \\ Y = 6 : \text{ STABLE} \end{array}}$$

② $xY' - Y = 6x^4 - 16x^3 + 5$

1) $Y' - \frac{1}{x}Y = 6x^3 - 16x^2 + \frac{5}{x}$

2) $\underline{u(x)} = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = (e^{\ln x})^{-1} = \underline{x^{-1}}$

3) $x^{-1}\left[Y' - \frac{1}{x}Y\right] = x^{-1}\left[6x^3 - 16x^2 + \frac{5}{x}\right]$

$(x^{-1}Y)' = 6x^2 - 16x + \frac{5}{x^2}$

4) $\underline{x^{-1}Y} = \int (6x^2 - 16x + 5x^{-2})\,dx = 2x^3 - 8x^2 - 5x^{-1} + C$

$\underline{Y = 2x^4 - 8x^3 - 5 + Cx}$ $\qquad$ WHEN $x = 1$, $Y = 0$; SO $2 - 8 - 5 + C = 0$ AND $\underline{C = 11}$

$$\boxed{Y = 2x^4 - 8x^3 - 5 + 11x}$$

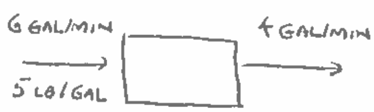
③ $x + Y - 2z = 5$
$2x + 3Y - 8z = 12$
$x - Y + 6z = 1$

$$\begin{bmatrix} 1 & 1 & -2 & 5 \\ 2 & 3 & -8 & 12 \\ 1 & -1 & 6 & 1 \end{bmatrix} \xrightarrow[-R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & 1 & -2 & 5 \\ 0 & 1 & -4 & 2 \\ 0 & -2 & 8 & -4 \end{bmatrix} \xrightarrow[2R_2+R_3]{-R_2+R_1} \begin{array}{c} \begin{array}{ccc} x & Y & z \end{array} \\ \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & -4 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

$x + 2z = 3$
$Y - 4z = 2$

LET $\underline{z = T}$ : $\boxed{x = 3 - 2T,\ Y = 2 + 4T,\ z = T}$, $T$ IN $\mathbb{R}$

④




$$\boxed{\frac{dA}{dT} = 6(5) - 4\left(\frac{A}{425 + 2T}\right)}$$

NET GAIN:
2 GAL/MIN

⑤ $A^{-1} = \begin{bmatrix} 6 & 2 \\ 1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 4 & 5 \\ 2 & 3 \end{bmatrix}$

$$(A^TC)^{-1} = C^{-1}(A^T)^{-1} = C^{-1}(A^{-1})^T = \frac{1}{2}\begin{bmatrix} 3 & -5 \\ -2 & 4 \end{bmatrix}\begin{bmatrix} 6 & 1 \\ 2 & 3 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 8 & -12 \\ -4 & 10 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & -6 \\ -2 & 5 \end{bmatrix}}$$

⑥ $\underline{N(T)} = \dfrac{K}{1 + ae^{-rT}} = \underline{\dfrac{1500}{1 + ae^{-rT}}}$

1) $\underline{T = 0}$ : $\dfrac{1500}{1 + a \cdot 1} = 300$ SO $\dfrac{1500}{300} = 1 + a$, $1 + a = 5$, $\underline{a = 4}$ AND $\underline{N = \dfrac{1500}{1 + 4e^{-rT}}}$

2) $\underline{T = 11}$ : $\dfrac{1500}{1 + 4e^{-11r}} = 500$ SO $\dfrac{1500}{500} = 1 + 4e^{-11r}$, $3 = 1 + 4e^{-11r}$, $4e^{-11r} = 2$,

$e^{-11r} = \frac{1}{2}$, $(e^{-11r})^{\frac{1}{11}} = \left(\frac{1}{2}\right)^{\frac{1}{11}}$, $\underline{e^{-r} = \left(\frac{1}{2}\right)^{\frac{1}{11}}}$ SO $\underline{N = \dfrac{1500}{1 + 4\left(\frac{1}{2}\right)^{T/11}}}$

(OR USE $\ln e^{-11r} = \ln\frac{1}{2}$, $-11r = \ln\frac{1}{2}$, $\underline{-r = \frac{1}{11}\ln\frac{1}{2}}$)

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6 (CONTINUED)

3) IF $N = 900$, $\dfrac{1500}{1+4(\frac{1}{2})^{T/11}} = 900$ so $\dfrac{1500}{900} = 1 + 4(\tfrac{1}{2})^{T/11}$, $\tfrac{5}{3} = 1 + 4(\tfrac{1}{2})^{T/11}$,

$\tfrac{2}{3} = 4(\tfrac{1}{2})^{T/11}$, $(\tfrac{1}{2})^{T/11} = \tfrac{1}{6}$, $\ln(\tfrac{1}{2})^{T/11} = \ln\tfrac{1}{6}$, $\tfrac{T}{11}\ln\tfrac{1}{2} = \ln\tfrac{1}{6}$,

$T = \boxed{\dfrac{11\ln\frac{1}{6}}{\ln\frac{1}{2}}\text{ mo.}} = \boxed{\dfrac{11\ln 6}{\ln 2}\text{ mo.}}$ (SINCE $\ln\frac{1}{x} = -\ln x$)

7 $A = \begin{bmatrix} 4 & 4 \\ 7 & 1 \end{bmatrix}$

$\det(A - \lambda I) = \begin{vmatrix} 4-\lambda & 4 \\ 7 & 1-\lambda \end{vmatrix} = (4-\lambda)(1-\lambda) - 28 = \lambda^2 - 5\lambda - 24 = 0$

$(\lambda - 8)(\lambda + 3) = 0$ SO $\boxed{\lambda = 8}$, $\boxed{\lambda = -3}$ ARE THE EIGENVALUES.

1) $\underline{\lambda = 8}$: SOLVING $(A - 8I)x = 0$ GIVES

$\begin{bmatrix} -4 & 4 & 0 \\ 7 & -7 & 0 \end{bmatrix} \xrightarrow[-7R_1+R_2]{-\frac{1}{4}R_1} \begin{array}{ccc} x & y & \\ \end{array} \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ LET $\underline{y = t}$, SO $\underline{x = t}$ AND

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} t \\ t \end{bmatrix} = t\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, AND $\boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$ IS AN EIGENVECTOR FOR $\lambda = 8$.

2) $\underline{\lambda = -3}$: SOLVING $(A + 3I)x = 0$ GIVES

$\begin{bmatrix} 7 & 4 & 0 \\ 7 & 4 & 0 \end{bmatrix} \xrightarrow[\frac{1}{7}R_1]{-R_1+R_2} \begin{array}{ccc} x & y & \\ \end{array} \begin{bmatrix} 1 & 4/7 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ LET $\underline{y = 7t}$, SO $\underline{x = -4t}$ AND

$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -4t \\ 7t \end{bmatrix} = t\begin{bmatrix} -4 \\ 7 \end{bmatrix}$, AND $\boxed{\begin{bmatrix} -4 \\ 7 \end{bmatrix}}$ IS AN EIGENVECTOR FOR $\lambda = -3$.

8 $\dfrac{dy}{dt} = y^3 + 3y^2 - 6y - 8$ $\qquad g'(y) = 3y^2 + 6y - 6$

1) $g'(2) = 12 + 12 - 6 = 18 > 0$, SO $\boxed{y = 2 \text{ IS UNSTABLE}}$

2) $g'(-1) = 3 - 6 - 6 = -9 < 0$, SO $\boxed{y = -1 \text{ IS STABLE}}$

9 $N(1) = LN(0) = \begin{bmatrix} 2 & 3 & 1.5 \\ .6 & 0 & 0 \\ 0 & .8 & 0 \end{bmatrix}\begin{bmatrix} 40 \\ 25 \\ 20 \end{bmatrix} = \boxed{\begin{bmatrix} 185 \\ 24 \\ 20 \end{bmatrix}}$

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(10) $\left[\begin{array}{ccc|ccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 1 & 3 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{-R_1+R_3} \left[\begin{array}{cccccc} 1 & 2 & 4 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow[-R_2+R_3]{-2R_2+R_1} \left[\begin{array}{cccccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & -1 & -1 & 1 \end{array}\right]$

$\xrightarrow[-R_3]{2R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & -2 & 0 \\ 0 & 1 & 0 & -2 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 & -1 \end{array}\right]$ so $A^{-1} = \boxed{\begin{bmatrix} 1 & -2 & 0 \\ -2 & -1 & 2 \\ 1 & 1 & -1 \end{bmatrix}}$

REMARK we can check this answer by verifying that $AA^{-1} = I$ (or that $A^{-1}A = I$).

(11) $\dfrac{dw}{dt} = k\sqrt{w}$

$\displaystyle\int \frac{1}{\sqrt{w}}\,dw = \int k\,dt$ $\quad \displaystyle\int w^{-1/2}\,dw = kt + C$ $\quad 2w^{1/2} = kt + C$

$\sqrt{w} = \ell t + D$ so $w = (\ell t + D)^2$

1) when $t=0$, $w=9$: $\sqrt{9} = 0 + D$ so $D = 3$

2) when $t=10$, $w=81$: $\sqrt{81} = 10\ell + 3$ so $10\ell + 3 = 9$, $10\ell = 6$, $\ell = \frac{3}{5}$

Then $w = \left(\frac{3}{5}t + 3\right)^2$,

so $w = 144$ gives $\left(\frac{3}{5}t + 3\right)^2 = 144$, $\frac{3}{5}t + 3 = 12$, $\frac{3}{5}t = 9$, $\boxed{t = 15}$

(since $\frac{3}{5}t + 3 > 0$)