MATH 17B SEC. 7.3 - PARTIAL FRACTIONS

TO FIND AN INTEGRAL OF THE FORM $\int \frac{P(x)}{Q(x)}\,dx$ WHERE $P(x)$ AND $Q(x)$ ARE POLYNOMIALS,

① DIVIDE $P(x)$ BY $Q(x)$ IF $\deg P(x) \geq \deg Q(x)$

② FACTOR $Q(x)$ INTO A PRODUCT OF POWERS OF 1ST-DEGREE POLYNOMIALS AND IRREDUCIBLE QUADRATICS. [ax^2+bx+c IS IRREDUCIBLE IFF $b^2-4ac<0$).

③ WRITE $\frac{P(x)}{Q(x)}$ AS A SUM OF PARTIAL FRACTIONS:

i) EACH FACTOR OF $Q(x)$ OF THE FORM $(ax+b)^n$ GIVES A SUM OF TERMS

$$\frac{c_1}{ax+b} + \frac{c_2}{(ax+b)^2} + \cdots + \frac{c_n}{(ax+b)^n}, \quad \text{AND}$$

ii) EACH FACTOR OF $Q(x)$ OF THE FORM $(ax^2+bx+c)^n$ GIVES A SUM OF TERMS

$$\frac{d_1x+e_1}{ax^2+bx+c} + \frac{d_2x+e_2}{(ax^2+bx+c)^2} + \cdots + \frac{d_nx+e_n}{(ax^2+bx+c)^n}.$$

EXAMPLES

a) $$\frac{9x-2}{x^3(x-1)^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x-1} + \frac{E}{(x-1)^2}$$

b) $$\frac{x^2+x+18}{x(x+8)^2(x^2+9)^2} = \frac{A}{x} + \frac{B}{x+8} + \frac{C}{(x+8)^2} + \frac{Dx+E}{x^2+9} + \frac{Fx+G}{(x^2+9)^2}$$

c) $$\frac{7x-5}{(x-4)^2(x^2+2x+5)} = \frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{Cx+D}{x^2+2x+5}$$

④ SOLVE FOR THE COEFFICIENTS BY MULTIPLYING THROUGH BY $Q(x)$ AND THEN

A) SUBSTITUTING VALUES OF x FOR WHICH $Q(x)=0$ AND/OR

B) EQUATING COEFFICIENTS OF LIKE POWERS OF x ON BOTH SIDES.

⑤ INTEGRATE EACH PARTIAL FRACTION.

EX FIND $\int \frac{5x+2}{x^3+4x}\,dx$.

$$\frac{5x+2}{x^3+4x} = \frac{5x+2}{x(x^2+4)} = \frac{A}{x} + \frac{Bx+C}{x^2+4}$$

$$5x+2 = A(x^2+4) + (Bx+C)x = (A+B)x^2 + Cx + 4A$$

$x=0$: $2 = 4A$ so $A = 1/2$

COEFF. OF x: $5 = C$

COEFF. OF x^2: $0 = A+B = \frac{1}{2} + B$ so $B = -\frac{1}{2}$

$$\int \frac{5x+2}{x^3+4x}\,dx = \int \left(\frac{1/2}{x} + \frac{-\frac{1}{2}x+5}{x^2+4} \right) dx = \frac{1}{2}\int \frac{1}{x}\,dx - \frac{1}{2}\cdot\frac{1}{2}\int \frac{2x}{x^2+4}\,dx + 5\int \frac{1}{x^2+4}\,dx$$

$$= \boxed{\frac{1}{2}\ln|x| - \frac{1}{4}\ln(x^2+4) + 5\cdot\frac{1}{2}\arctan\frac{x}{2} + C}$$