**Math 178**

**Sec. 6.3 - Applications of the Definite Integral**

1. **Area**

   a) \[ A = \int_a^b (f(x) - g(x)) \, dx \]

   b) \[ A = \int_c^d (f(y) - g(y)) \, dy \]

2. **Average Value**

   \[ f_{\text{avg}} = \frac{1}{b-a} \int_a^b f(x) \, dx \]

3. **Volume**

   a) \[ V = \int_a^b \pi \left( (f(x))^2 - (g(x))^2 \right) \, dx \]

   b) \[ V = \int_c^d \pi \left( (f(y))^2 - (g(y))^2 \right) \, dy \]

4. **Arc Length**

   \[ s = \int_a^b \sqrt{1 + (f'(x))^2} \, dx \]

5. **Displacement and Distance**

   If \( s(t) \) is the position function for an object moving along a line,

   a) The displacement for \( a \leq t \leq b \) is given by

   \[ s(b) - s(a) = \int_a^b s'(t) \, dt \]

   b) The distance traveled for \( a \leq t \leq b \) is given by

   \[ d(b) - d(a) = \int_a^b |s'(t)| \, dt \]
Suppose we want to determine how blood flow through an artery changes when its radius changes,

\[ a \downarrow \quad \text{We can think of the artery as a cylinder with radius } a. \]

Due to friction, the velocity of the blood is less near the walls of the artery.

The French physician Poiseuille discovered in about 1840 that the velocity of the blood is related to the distance \( r \) from the center of the artery by the formula

\[ V(r) = \frac{k(a^2 - r^2)}{r} \]

where \( k \) is a constant.

To determine the volume of blood which flows through a cross-section of the artery per unit time, denoted by \( F \) (for flux), we need to cut the cross-section into circular rings, since the velocity is approximately constant on a thin circular ring:

For a circular ring with inner radius \( r_{i-1} \) and outer radius \( r_i \), the volume of blood \( F_i \) which flows across the ring per unit time is approximated by

\[ F_i = (\text{velocity}) \times (\text{area}) \approx V(r_i) \left( a \pi r_i \Delta r_i \right) \]

(since area = \( \pi r_i^2 - \pi r_{i-1}^2 = \pi (r_i + r_{i-1}) (r_i - r_{i-1}) \approx 2 \pi r_i \Delta r_i \)).

Therefore, the total flux \( F \) is given by

\[ F = \lim_{\Delta r_i \to 0} \sum_{i=1}^{n} 2 \pi r_i V(r_i) \Delta r_i = \int_0^a 2 \pi r V(r) \, dr = \int_0^a 2 \pi r \left( k \left( a^2 - r^2 \right) \right) \, dr \]

\[ = 2 \pi k \left[ \frac{a^2 r^2}{2} - \frac{r^4}{4} \right]_0^a = 2 \pi k \left( \frac{a^4}{2} - \frac{a^4}{4} \right) = \pi k a^4. \]

This shows that the blood flow is proportional to the 4th power of the radius.

Remarks

(1) By Ex. 5 in Sec. 4.8, increasing the radius of the artery by \( \frac{a}{2} \) increases the blood flow by about \( 4 \frac{a}{2} \) (for small values of \( r \)).

(2) If a doctor uses angioplasty to increase the radius of a closed artery by 10\%, then the blood flow increases by about \( 16 \frac{a}{2} \) (since \((1.1)^4 = 1.4641\)).

(3) The above discussion is adapted from Stewart's Calculus textbook, and further information about Poiseuille's Law can be found at [www.math.arizona.edu/~maw1999/blood/poiseuille/](http://www.math.arizona.edu/~maw1999/blood/poiseuille/).

(4) More generally, Poiseuille's Law can be used to model the flow of fluids through cylindrical pipes.