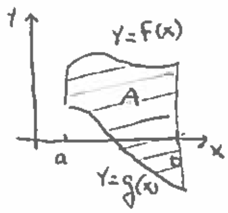
MATH 17B

Sec. 6.3 – Applications of the Definite Integral

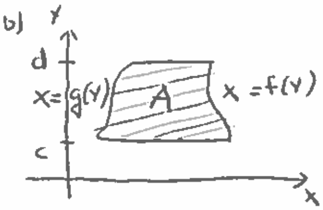
① AREA

a)



$$A = \int_a^b \left(f(x) - g(x)\right) dx$$

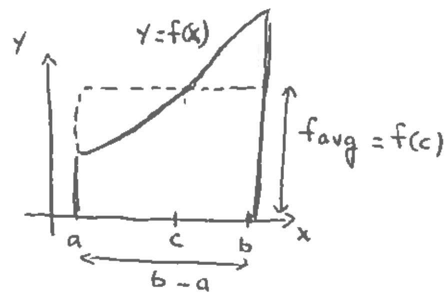
b)



$$A = \int_c^d \left(f(y) - g(y)\right) dy$$

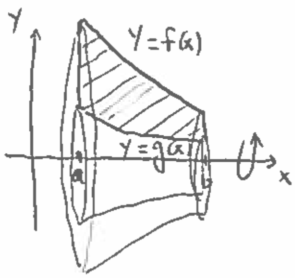
② AVERAGE VALUE

$$f_{avg} = \frac{1}{b-a} \int_a^b f(x)\, dx$$



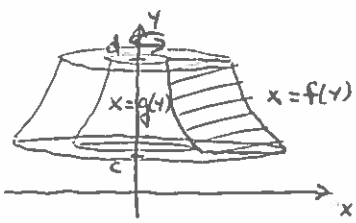
③ VOLUME

a)



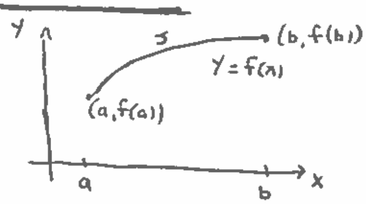
$$V = \int_a^b \pi\left((f(x))^2 - (g(x))^2\right) dx$$

b)



$$V = \int_c^d \pi\left((f(y))^2 - (g(y))^2\right) dy$$

④ ARCLENGTH



$$s = \int_a^b \sqrt{1 + (f'(x))^2}\, dx$$

⑤ DISPLACEMENT AND DISTANCE

If $s(t)$ is the position function for an object moving along a line,

a) The <u>displacement</u> for $a \le t \le b$ is given by

$$s(b) - s(a) = \int_a^b v(t)\, dt$$

, and

b) The <u>distance</u> traveled for $a \le t \le b$ is given by

$$d(b) - d(a) = \int_a^b |v(t)|\, dt$$

MATH 17B

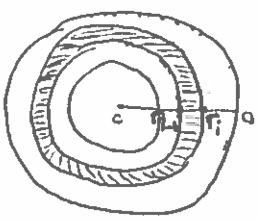
Sec. 6.3 – An Application of Integration to Medicine

Suppose we want to determine how blood flow through an artery changes when its radius changes.

We can think of the artery as a cylinder with radius a. Due to friction, the velocity of the blood is less near the walls of the artery:

The French physician Poiseuille discovered in about 1840 that the velocity of the blood is related to the distance r from the center of the artery by the formula $\underline{v(r) = k(a^2 - r^2)}$, where k is a constant.

To determine the volume of blood which flows through a cross-section of the artery per unit time, denoted by F (for $\underline{\text{flux}}$), we need to cut the cross-section into circular rings, since the velocity is approximately constant on a thin circular ring:



For a circular ring with inner radius r_{i-1} and outer radius r_i, the volume of blood F_i which flows across the ring per unit time is approximated by

$$\underline{F_i} = (\text{velocity}) \times (\text{area}) \approx \underline{v(r_i)\,(2\pi r_i \,\Delta r_i)}$$

$$\left(\text{since area} = \pi r_i^2 - \pi r_{i-1}^2 = \pi(r_i + r_{i-1})\underline{(r_i - r_{i-1})} \approx 2\pi r_i \underline{\Delta r_i}\right).$$

Therefore the total flux F is given by

$$\underline{F} = \lim_{\|P\| \to 0} \sum_{i=1}^{n} 2\pi r_i\, v(r_i)\,\Delta r_i = \int_0^a 2\pi r\, \underline{v(r)}\, dr = \int_0^a 2\pi r\,\underline{(k(a^2 - r^2))}\, dr$$

$$= 2\pi k \int_0^a (a^2 r - r^3)\, dr = 2\pi k \left[a^2 \cdot \frac{r^2}{2} - \frac{r^4}{4}\right]_0^a = 2\pi k\left(\frac{a^4}{2} - \frac{a^4}{4}\right) = 2\pi k \cdot \frac{a^4}{4} = \boxed{\frac{\pi}{2} k a^4}.$$

This shows that <u>the blood flow is proportional to the 4th power of the radius.</u>

<u>Remarks</u>

A) By Ex. 5 in Sec. 4.8, increasing the radius of the artery by $p\%$ increases the blood flow by about $4p\%$ (for small values of p).

B) If a doctor uses angioplasty to increase the radius of a clogged artery by 10%, then the blood flow increases by about 46% (since $(1.1)^4 = 1.4641$).

C) The above discussion is adapted from Stewart's Calculus text, and further information about Poiseuille's law can be found at www.math.arizona.edu/~maw1999/blood/poiseuille/.

D) More generally, Poiseuille's law can be used to model the flow of fluids through cylindrical pipes.