Sec. 9.2 – Determinants of $n \times n$ Matrices (Optional)

Let $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$ be an $n \times n$ matrix.

DEF The cofactor C_{ij} of the entry a_{ij} is given by $C_{ij} = (-1)^{i+j} M_{ij}$,

where M_{ij} is the determinant of the matrix obtained from A by deleting the ith row and jth column.

EX If $A = \begin{bmatrix} 5 & 2 & 7 \\ 3 & 6 & 4 \\ 9 & 1 & -5 \end{bmatrix}$, $C_{12} = (-1)^3 \begin{vmatrix} 3 & 4 \\ 9 & -5 \end{vmatrix} = 51$ and $C_{31} = (-1)^4 \begin{vmatrix} 2 & 7 \\ 6 & 4 \end{vmatrix} = -34$

DEF The determinant of A is given by

$$\boxed{\det(A) = a_{i1} C_{i1} + a_{i2} C_{i2} + \cdots + a_{in} C_{in}} \quad \text{(expanding along the } i\text{th row)}$$

or

$$\boxed{\det(A) = a_{1j} C_{1j} + a_{2j} C_{2j} + \cdots + a_{nj} C_{nj}} \quad \text{(expanding along the } j\text{th column)}$$

REMARK Notice that the determinant can be found by selecting any row or column, multiplying the entries in that row or column by their cofactors, and then adding the resulting products.

EX (A) If $A = \begin{bmatrix} 2 & 9 & 4 \\ 1 & 5 & 3 \\ 6 & -1 & -2 \end{bmatrix}$, find $\det(A)$.

$$\det(A) = 1\left(-\begin{vmatrix} 9 & 4 \\ -1 & -2 \end{vmatrix}\right) + 5\left(\begin{vmatrix} 2 & 4 \\ 6 & -2 \end{vmatrix}\right) + 3\left(-\begin{vmatrix} 2 & 9 \\ 6 & -1 \end{vmatrix}\right) = 1(14) + 5(-28) + 3(56) = \boxed{42} \quad \text{(using row 2)}$$

or

$$\det(A) = 2\left(\begin{vmatrix} 5 & 3 \\ -1 & -2 \end{vmatrix}\right) + 1\left(-\begin{vmatrix} 9 & 4 \\ -1 & -2 \end{vmatrix}\right) + 6\left(\begin{vmatrix} 9 & 4 \\ 5 & 3 \end{vmatrix}\right) = 2(-7) + 1(14) + 6(7) = \boxed{42} \quad \text{(using column 1)}$$

(B) If $A = \begin{bmatrix} 3 & 2 & 5 \\ 4 & 1 & -1 \\ 6 & -3 & 2 \end{bmatrix}$, find $\det(A)$.

$$\det(A) = 3\left(\begin{vmatrix} 1 & -1 \\ -3 & 2 \end{vmatrix}\right) + 2\left(-\begin{vmatrix} 4 & -1 \\ 6 & 2 \end{vmatrix}\right) + 5\left(\begin{vmatrix} 4 & 1 \\ 6 & -3 \end{vmatrix}\right) = 3(-1) + 2(-14) + 5(-18) = \boxed{-121} \quad \text{(using row 1)}$$

or

$$\det(A) = 4\left(-\begin{vmatrix} 2 & 5 \\ -3 & 2 \end{vmatrix}\right) + 1\left(\begin{vmatrix} 3 & 5 \\ 6 & 2 \end{vmatrix}\right) + (-1)\left(-\begin{vmatrix} 3 & 2 \\ 6 & -3 \end{vmatrix}\right) = 4(-19) + 1(-24) + (-1)(21) = \boxed{-121} \quad \text{(using row 2)}$$

(C) If $A = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & -1 \\ 4 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 \end{bmatrix}$, find $\det(A)$.

$$\det(A) = 4\left(\begin{vmatrix} 0 & 2 & 1 \\ 1 & 3 & -1 \\ 2 & 1 & 1 \end{vmatrix}\right) + 1\left(\begin{vmatrix} 1 & 0 & 1 \\ 2 & 1 & -1 \\ 1 & 2 & 1 \end{vmatrix}\right) = 4\left[2\left(-\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\right) + 1\left(\begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix}\right)\right] +$$

$$1\left[1\left(\begin{vmatrix} 1 & -1 \\ 2 & 1 \end{vmatrix}\right) + 1\left(\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix}\right)\right] = 4\left[2(-3) + (-5)\right] + \left[3 + 3\right] = -44 + 6 = \boxed{-38}$$