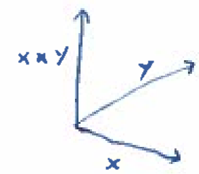
Sec. 9.4 – Cross-Products in $\mathbb{R}^3$

Let $x = \langle x_1, x_2, x_3 \rangle$ and $y = \langle y_1, y_2, y_3 \rangle$ be vectors in $\mathbb{R}^3$.

We can find a vector which is orthogonal to both x and y by taking their cross-product $x \times y$, which is defined by



$$\boxed{\underline{x \times y} = \begin{vmatrix} i & j & k \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} = \begin{vmatrix} x_2 & x_3 \\ y_2 & y_3 \end{vmatrix} i - \begin{vmatrix} x_1 & x_3 \\ y_1 & y_3 \end{vmatrix} j + \begin{vmatrix} x_1 & x_2 \\ y_1 & y_2 \end{vmatrix} k}$$

where $i = \langle 1, 0, 0 \rangle$, $j = \langle 0, 1, 0 \rangle$, and $k = \langle 0, 0, 1 \rangle$.

Ex A) If $x = \langle 1, 1, 3 \rangle$ and $y = \langle 1, 2, 4 \rangle$,

$$\underline{x \times y} = \begin{vmatrix} i & j & k \\ 1 & 1 & 3 \\ 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 2 & 4 \end{vmatrix} i - \begin{vmatrix} 1 & 3 \\ 1 & 4 \end{vmatrix} j + \begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix} k = \langle -2, -1, 1 \rangle$$

B) If $x = \langle 5, 2, 1 \rangle$ and $y = \langle 3, 4, 6 \rangle$,

$$\underline{x \times y} = \begin{vmatrix} i & j & k \\ 5 & 2 & 1 \\ 3 & 4 & 6 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 4 & 6 \end{vmatrix} i - \begin{vmatrix} 5 & 1 \\ 3 & 6 \end{vmatrix} j + \begin{vmatrix} 5 & 2 \\ 3 & 4 \end{vmatrix} k = \langle 8, -27, 14 \rangle$$

C) If $x = \langle 1, 7, 2 \rangle$ and $y = \langle 4, 3, -1 \rangle$,

$$\underline{x \times y} = \begin{vmatrix} i & j & k \\ 1 & 7 & 2 \\ 4 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 7 & 2 \\ 3 & -1 \end{vmatrix} i - \begin{vmatrix} 1 & 2 \\ 4 & -1 \end{vmatrix} j + \begin{vmatrix} 1 & 7 \\ 4 & 3 \end{vmatrix} k = \langle -13, 9, -25 \rangle$$

Remarks

1) We can check that $x \times y$ is orthogonal to x and y by seeing if

$\underline{x \cdot (x \times y) = 0}$ and $\underline{y \cdot (x \times y) = 0}$.

2) $\underline{y \times x = -(x \times y)}$, and $x \times y = \vec{0}$ iff x and y are parallel.

3) The direction of $x \times y$ is determined by the right-hand rule, and its length is given by $\underline{|x \times y| = |x||y| \sin\theta}$ (where θ is the angle between x and y).

[Notice that this is the area of the parallelogram determined by x and y.]

4) If A, B, and C are noncollinear points in a plane and $x = \overrightarrow{AB}$ and $y = \overrightarrow{AC}$, then $x \times y$ gives a normal vector to the plane.