9.1 - Solving Linear Systems

DEF A system of linear equations (or linear system) is a set of equations

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \qquad (m \text{ equations, } n \text{ unknowns})$$
$$\vdots \qquad\qquad \vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Ex A) $2x - y = 5$, $x + y = 4$ (2 lines) B) $x - y + 2z = 5$, $2x + y + 6z = 12$ (2 planes)

REMARK In general, a linear system can either have 1) no solution 2) exactly one solution or 3) infinitely many solutions

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DEF A linear system is consistent if it has a solution, and inconsistent if it has no solution.

Ex 1 Solve the linear system

$$x + y + 7z = 5$$
$$x + 2y + 4z = -2$$
$$2x + 5y + 6z = -10.$$

We'll start with the augmented matrix $\left[\begin{array}{ccc|c} 1 & 1 & 7 & 5 \\ 1 & 2 & 4 & -2 \\ 2 & 5 & 6 & -10 \end{array}\right]$.

Our goal is to change this matrix so that it has the following properties:

a) Each nonzero row has a 1 as its first nonzero entry (called a leading 1).
b) The leading 1's go from left to right, reading down the matrix.
c) Any column with a leading 1 has zeros everywhere else.
d) Zero rows are at the bottom.

A matrix is in reduced row echelon form if it has these properties.

To get our matrix in this form, we will use the following operations:

Elementary Row Operations

1. Add a multiple of one row to another row.
2. Multiply a row by a nonzero number.
3. Interchange two rows.

Finishing this example, we have

$$\left[\begin{array}{ccc|c} 1 & 1 & 7 & 5 \\ 1 & 2 & 4 & -2 \\ 2 & 5 & 6 & -10 \end{array}\right] \xrightarrow[-2R_1+R_3]{-R_1+R_2} \left[\begin{array}{ccc|c} 1 & 1 & 7 & 5 \\ 0 & 1 & -3 & -7 \\ 0 & 3 & -8 & -20 \end{array}\right] \xrightarrow[-R_2+R_1]{-3R_2+R_3} \left[\begin{array}{ccc|c} 1 & 0 & 10 & 12 \\ 0 & 1 & -3 & -7 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow[-10R_3+R_1]{3R_3+R_2} \begin{array}{c} \begin{array}{ccc} x & y & z \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \end{array}\right] \end{array}$$

This gives the solution $\boxed{\begin{array}{l} x = 2 \\ y = -4 \\ z = 1 \end{array}}$.

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EX 2 – SOLVE
$$x + y + z = 8$$
$$2x - y - 4z = 7.$$

$$\begin{bmatrix} 1 & 1 & 1 & 8 \\ 2 & -1 & -4 & 7 \end{bmatrix} \xrightarrow{-2R_1+R_2} \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & -3 & -6 & -9 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 1 & 1 & 8 \\ 0 & 1 & 2 & 3 \end{bmatrix} \xrightarrow{-R_2+R_1} \begin{array}{c} \begin{array}{cccc} x & y & z & \end{array} \\ \left[\begin{array}{ccc|c} 1 & 0 & -1 & 5 \\ 0 & 1 & 2 & 3 \end{array}\right] \end{array}$$

SINCE z DOES NOT CORRESPOND TO A LEADING 1, IT IS A FREE VARIABLE:

LET $z = t$, SO $x - t = 5$ (ROW 1), $y + 2t = 3$ (ROW 2) GIVES
$$\boxed{\begin{aligned} x &= 5 + t \\ y &= 3 - 2t \\ z &= t \end{aligned}}$$
(WHERE t IS ANY REAL NUMBER)

EX 3 – SOLVE
$$x + 2y - z + 4w = 13$$
$$x - y + 2z - 5w = 1$$
$$2x + 3y - z + 5w = 22.$$

$$\begin{bmatrix} 1 & 2 & -1 & 4 & 13 \\ 1 & -1 & 2 & -5 & 1 \\ 2 & 3 & -1 & 5 & 22 \end{bmatrix} \xrightarrow[-2R_1+R_3]{-R_1+R_2} \begin{bmatrix} 1 & 2 & -1 & 4 & 13 \\ 0 & -3 & 3 & -9 & -12 \\ 0 & -1 & 1 & -3 & -4 \end{bmatrix} \xrightarrow{-\frac{1}{3}R_2} \begin{bmatrix} 1 & 2 & -1 & 4 & 13 \\ 0 & 1 & -1 & 3 & 4 \\ 0 & -1 & 1 & -3 & -4 \end{bmatrix}$$

$$\xrightarrow[-2R_2+R_1]{R_2+R_3} \begin{array}{c} \begin{array}{ccccc} x & y & z & w & \end{array} \\ \left[\begin{array}{cccc|c} 1 & 0 & 1 & -2 & 5 \\ 0 & 1 & -1 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \end{array}$$

HERE z AND w ARE FREE VARIABLES:

LET $z = s$, $w = t$: $x + s - 2t = 5$ (ROW 1), $y - s + 3t = 4$ (ROW 2) SO
$$\boxed{\begin{aligned} x &= 5 - s + 2t \\ y &= 4 + s - 3t \\ z &= s \\ w &= t \end{aligned}}$$
(WHERE s AND t ARE ANY REAL NUMBERS)

EX 4 – SOLVE
$$x - 2y + z = 4$$
$$2x - y - z = 3$$
$$5x - 4y - z = 7.$$

$$\begin{bmatrix} 1 & -2 & 1 & 4 \\ 2 & -1 & -1 & 3 \\ 5 & -4 & -1 & 7 \end{bmatrix} \xrightarrow[-5R_1+R_3]{-2R_1+R_2} \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 3 & -3 & -5 \\ 0 & 6 & -6 & -13 \end{bmatrix} \xrightarrow{\frac{1}{3}R_2} \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -5/3 \\ 0 & 6 & -6 & -13 \end{bmatrix}$$

$$\xrightarrow{-6R_2+R_3} \begin{bmatrix} 1 & -2 & 1 & 4 \\ 0 & 1 & -1 & -5/3 \\ 0 & 0 & 0 & -3 \end{bmatrix}$$

$\longleftarrow$ THIS GIVES $0 = -3$, SO THERE IS $\boxed{\text{NO SOLUTION}}$

(THE SYSTEM IS INCONSISTENT)