4 If \( \gamma: (a,b) \to C \) is differentiable and \( f: A \to C \) is analytic with \( f([a,b]) \subseteq A \),
then \( \sigma = \gamma \circ \delta \) is differentiable with \( \sigma'(t) = f'(\gamma(t))\gamma'(t) \).

Let \( t \in (a,b) \), and define \( h: A \to C \) by
\[
h(w) = \begin{cases} \frac{f(w) - f(\gamma(t))}{w - \gamma(t)}, & w \neq \gamma(t) \\ f'(\gamma(t)), & w = \gamma(t) \end{cases}
\]
Then \( h \) is continuous at \( \gamma(t) \), since
\[
\lim_{w \to \gamma(t)} h(w) = f'(\gamma(t)) = h'(\gamma(t)) \Rightarrow h \) is continuous at \( \gamma(t) \).
Then \( \sigma'(t) - \sigma'(t) = h'(\gamma(t)) \gamma'(t) - \sigma'(t) = \sigma'(t) - \gamma'(t) \gamma'(t) \) for \( t \neq \gamma(t) \), since
\[
\frac{\sigma'(t) - \sigma'(t)}{t - \gamma(t)} = \frac{f'(\gamma(t)) - f'(\gamma(t))}{t - \gamma(t)} \gamma'(t) = \gamma'(t) - \gamma'(t) = \gamma'(t) - \gamma'(t) \),
and the result holds if \( \gamma(t) = \gamma'(t) \).
So \( \sigma'(t) = \lim_{r \to t} \sigma'(r) \), \( \gamma'(t) = \lim_{r \to t} \gamma'(r) \),
\[
\sigma'(t) = \left( \lim_{r \to t} h'(\gamma(r)) \right) \gamma'(t) = \left( \lim_{r \to t} \frac{f'(\gamma(r))}{\gamma'(r)} \right) \gamma'(t) = h'(\gamma(t)) \gamma'(t).
\]

5 b) \( f(z) = z^6 + 3z^3, z = 0 \)
\[
f'(z) = 6z^5 + 3, \quad f'(0) = 3 \neq 0 \Rightarrow \text{TRANSPARENT VECTORS AT } 0 \text{ THROUGH AN ANGLE OF } \theta = 0 \text{ AND STRETCHES THEM BY A FACTOR OF } \gamma = 3.
\]
\[
c) f(z) = \frac{z^2 + z + 1}{z - 1}, z = 0
f'(z) = \frac{(z - 1)(2z + 1) - (z^2 + z + 1) \cdot 1}{(z - 1)^2}, \quad f'(0) = -\frac{1}{2} \neq \frac{2}{3} \Rightarrow \text{NO FIELD OF TANGENT VECTORS AT } 0 \text{ THROUGH AN ANGLE OF } \theta = \pi \text{ AND STRETCHES THEM BY A FACTOR OF } \gamma = 2.
\]

6 f(z) = \text{is not analytic at any point in } C.
\[
f(z) = \sqrt{z} \Rightarrow f(z) = \sqrt{x + iy}, \quad \text{so } f = u + iv \text{ where } u = \sqrt{x^2 + y^2} \text{ and } v = 0.
\]
Then \( u_x = \frac{x}{\sqrt{x^2 + y^2}}, \quad u_y = \frac{y}{\sqrt{x^2 + y^2}}, \quad \text{for } (x,y) \neq (0,0), \quad \text{and } u_x = 0 = u_y, \quad \text{so } \text{f is not differentiable at } z = 0 = \frac{z}{2} \Rightarrow \text{f is not analytic at any } z \in C.
\]

REMARC Notice that f is not differentiable at 0 since \( f'(0) = \left. \frac{f(z) - f(0)}{z - 0} \right|_{z \to 0} = \lim_{z \to 0} \frac{z}{z} \text{ does not exist!}
\]
\[
\quad i) \text{for } z \in \mathbb{R} \text{ with } z > 0, \quad \frac{z}{z} = 1 \text{ and } \frac{1}{z} = -1.
\]
\[
\quad i) \text{for } z \in \mathbb{R} \text{ with } z < 0, \quad \frac{z}{z} = -1 \text{ and } \frac{1}{z} = -1.
\]
19) \[ f(z) = \frac{z+1}{z-1} \]

a) \[ f \text{ is analytic on } C \setminus \{1\} \quad \text{(where } z \neq 1) \]

b) Since \[ f'(z) = -\frac{2}{(z-1)^2} \]
\[ f'(0) = -2 \neq 0 \] and hence \( f \) is conformal at 0.

c) i) Image of x-axis: if \( z = x + iy \), then \( f(z) = \frac{x+1}{x-1} \) and for \( x \in \mathbb{R} \),

\[ \frac{x+1}{x-1} = \frac{t}{t} \iff x = t \iff x = \frac{t+1}{t-1} \text{ and } t \neq 1. \]
Therefore, the image of the x-axis is the x-axis with 1 excluded:
\{ \( z \mid \frac{z}{z-1} \), \( y = 0 \) and \( x \neq \frac{1}{2} \). \]

ii) Image of y-axis: if \( z = iy \), then

\[ f(z) = \frac{y + i + 1}{y - 1 + iy - 1} = \frac{y^2 + 1 - 1}{y^2 + 1} \]

\[ u = \frac{y^2 - 1}{y^2 + 1} \text{ and } v = \frac{2y}{y^2 + 1} \]

\[ u^2 + v^2 = \frac{(y^2 - 1)^2 + (2y)^2}{(y^2 + 1)^2} = \frac{(y^2 + 1)^2}{(y^2 + 1)^2} = 1. \]

Therefore, the image lies on the unit circle.

If \( \cos \theta, \sin \theta \) is any point on the circle other than \((1,0)\),

solving \( \cos \theta = \frac{y^2 - 1}{y^2 + 1} \) gives \( y = \pm \frac{1 + \cos \theta}{1 - \cos \theta} \) (with the sign determined by the sign of \( \sin \theta \)).

So this gives a point \( z = iy \) which maps to \( (\cos \theta, \sin \theta) \).

Therefore, the image of the y-axis is the unit circle with the point \((1,0)\) deleted:
\{ \( z \mid |z| = 1 \) and \( z \neq \frac{1}{2} \). \}

b) Since \( f \) is conformal at 0 and the x-axis and y-axis intersect at an angle of \( \frac{\pi}{2} \), their images also intersect at an angle of \( \frac{\pi}{2} \).

20) If \( f \) is analytic on a region \( A \) and \( f^{(n+1)}(z) = 0 \) on \( A \),

then \( f \) is a polynomial of degree at most \( n \).

Proof by induction on \( n \):

1) This is true for \( n = 0 \), since \( f(z) = 0 \) on \( A \) implies that \( f \) is constant on \( A \).

2) Assume the statement is true for an integer \( n \), where \( n \geq 0 \),

and let \( f^{(n+1)}(z) = 0 \) on \( A \).

If \( g(z) = f^{(n+1)}(z) \), then \( g'(z) = 0 \) on \( A \) so \( g(z) = C \) on \( A \) for some constant \( C \).

If \( h(z) = f(z) - Cz^n \), then \( h^{(n+1)}(z) = f^{(n+1)}(z) - C = 0 \) on \( A \).

So by the induction hypothesis \( h(z) \) is a polynomial of degree at most \( n \),

Thus \( f(z) = h(z) + \frac{Cz^{n+1}}{(n+1)!} \) is a polynomial of degree at most \( n+1 \),

so the assertion is valid for \( n+1 \).

Remark: Here we are using that \( D^n \left( \frac{z^n}{n!} \right) = 1 \) for any \( n \in \mathbb{N} \),

which follows by induction.
1.6 (b) \( f(z) = \log(z+1) \) gives \( f'(z) = \frac{1}{z+1} \).

If we take the principal branch of \( \log z \), then \( f \) is analytic for \( z+1 \in L = \{ z: x \leq 0 \text{ and } y = 0 \} \), so

\[
\lim_{z \to 0} \frac{e^z - 1}{z} = \lim_{z \to 0} \frac{f(z) - f(0)}{z - 0}
\]

so \( \lim_{z \to 0} \frac{e^z - 1}{z} = f'(0) = e^0 = 1 \) since \( D_z(e^z) = e^z \).

(c) \( \lim_{z \to 1} \frac{\log z}{z - 1} = \lim_{z \to 1} \frac{f(z) - f(1)}{z - 1} \) for \( f(z) = \log z \),

so \( \lim_{z \to 1} \frac{\log z}{z - 1} = f'(1) = \frac{1}{1} = 1 \) since \( D_z(\log z) = \frac{1}{z} \).

Ch. 1: Let \( f \) be analytic on \( A \), and define \( g: A \to \mathbb{C} \) by \( g(z) = \overline{f(z)} \).

Let \( f = u + iv \), so \( u_x = v_y \) and \( u_y = -v_x \) since \( f \) is analytic on \( A \). Then \( g = u - iv = u + iv \) where \( v^* = -v \),

so \( g \) is analytic iff \( u_x = v^*_y \) and \( u_y = -v^*_x \)

iff \( u_x = -v_y \) and \( u_y = v_x \)

iff \( v_y = -v_x \) and \( -v_x = v_x \) iff \( v_y = 0 = v_x \),

then \( f' = f_x = u_x + iv_x = 0 \), so \( f \) is constant on \( A \) (assuming \( A \) is a connected open subset of \( \mathbb{C} \)).

(f) \( f(x + iy) = (x^2 + y^2) + i(x^2 + y^2) \).

Where does \( f'(z) \) exist?

Since \( u_x = 2x \) and \( v_y = 2y \)

and \( u_y = 2 \) and \( v_x = 2x \),

so \( f \) is differentiable only at \((-1,-1)\).

Remark: Notice that \( f \) is differentiable at \((-1,-1)\).

Since it has continuous first partials and satisfies the Cauchy–Riemann equations there.