Sec. 6.6 - Centroids

In the region $R$ is bounded above by $y = f(x)$ and below by $y = g(x)$ for $a \leq x \leq b$, the centroid $(\overline{x}, \overline{y})$ of $R$ is given by

\[
\overline{x} = \frac{\int_a^b x (f(x) - g(x)) \, dx}{\int_a^b (f(x) - g(x)) \, dx} \quad \text{and} \quad \overline{y} = \frac{\int_a^b \frac{1}{2} (f(x)^2 - g(x)^2) \, dx}{\int_a^b (f(x) - g(x)) \, dx}
\]

Local Approximations

\[
m = \delta A \approx \delta \int_a^b (f(c) - g(c)) \, dx
\]

\[
M_x = \delta x m \approx \delta \frac{1}{2} \int_a^b (f(c) + g(c)) \left[ f(c) - g(c) \right] \, dx = \delta \frac{1}{2} \int_a^b \left( f(c)^2 - g(c)^2 \right) \, dx
\]

\[
M_y = \delta y m \approx \delta \frac{1}{2} \int_a^b (f(c) + g(c)) \left[ f(c) - g(c) \right] \, dy = \delta \frac{1}{2} \int_a^b \left( f(c)^2 - g(c)^2 \right) \, dy
\]

Remark

If the density is not constant, then the factor $\delta(x)$ must be included in the integrals in Case 1, and the factor $\delta(y)$ must be included in the integrals in Case 2. (If the density $\delta$ is a function of $x$ and $y$, then double integration must be used, as in Ch. 15.)

Pappus's Theorem for Volumes

If a region $R$ is revolved around a line $L$ which does not intersect the interior of $R$, the volume of the resulting solid is given by

\[
V = A(2\pi \delta)
\]

where $A$ is the area of $R$ and $\delta$ is the distance from the centroid of $R$ to $L$.

Remarks

a) Notice that $2\pi \delta$ is the distance the centroid travels when it revolves around $L$.

b) This theorem follows from the facts that $V_x = \pi \delta M_x$ and $V_y = \pi \delta M_y$.

For a region in the first quadrant,