Sec. 10.4 – Summary

Comparison Test (CT)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive-term series.

1) If $a_n \le b_n$ for $n \ge N$ (for some N) and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.
2) If $a_n \ge b_n$ for $n \ge N$ (for some N) and $\sum_{n=1}^{\infty} b_n$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Remarks

A) $\sum_{n=1}^{\infty} b_n$ is the series to which $\sum_{n=1}^{\infty} a_n$ is being compared;
it will usually be a p-series or a geometric series.

B) Notice that the CT gives no conclusion
if $a_n \ge b_n$ and $\sum_{n=1}^{\infty} b_n$ converges, or if $a_n \le b_n$ and $\sum_{n=1}^{\infty} b_n$ diverges.

C) We have $\frac{A}{B} \le \frac{C}{D}$ iff $BD\left(\frac{A}{B}\right) \le BD\left(\frac{C}{D}\right)$ iff $AD \le BC$ when $B, D > 0$.

D) Notice that $\ln n < n^k$ for $n \ge N$ (for some N) if $k > 0$, since $\lim_{n\to\infty} \frac{\ln n}{n^k} = 0$ if $k > 0$.

E) If the required inequality isn't satisfied for n sufficiently large,
 i) multiply $\sum_{n=1}^{\infty} b_n$ by any C with $C > 1$ if $\sum_{n=1}^{\infty} b_n$ converges, and
 multiply $\sum_{n=1}^{\infty} b_n$ by any C with $0 < C < 1$ if $\sum_{n=1}^{\infty} b_n$ diverges; or
 ii) use the following test instead.

Limit Comparison Test (LCT)

Let $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ be positive-term series, and let $\lim_{n\to\infty} \frac{a_n}{b_n} = L$ where $0 \le L \le \infty$.

1) If $\sum_{n=1}^{\infty} b_n$ converges and $L \ne \infty$, then $\sum_{n=1}^{\infty} a_n$ converges.
2) If $\sum_{n=1}^{\infty} b_n$ diverges and $L \ne 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

Remarks

A) If $0 < L < \infty$, then $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ either both converge or both diverge.

B) Notice that the LCT gives no conclusion
if $\sum_{n=1}^{\infty} b_n$ converges and $L = \infty$, or if $\sum_{n=1}^{\infty} b_n$ diverges and $L = 0$.

C) The LCT follows from the CT.

Sec. 10.4 – EXAMPLES

A) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2-3n+10}$

COMPARE TO $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$, WHICH DIVERGES SINCE IT'S THE HARMONIC SERIES:

$$\frac{n}{n^2-3n+10} \geq \frac{1}{n} \quad \text{IFF} \quad n^2 \geq n^2-3n+10 \quad \text{IFF} \quad 3n \geq 10 \quad \text{IFF} \quad n \geq 4,$$

SO $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2-3n+10}$ $\boxed{\text{DIVERGES}}$ BY THE COMPARISON TEST.

B) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{n^4+8n^2-5}$

COMPARE TO $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P>1$:

$$\frac{n^2+7}{n^4+8n^2-5} \leq \frac{1}{n^2} \quad \text{IFF} \quad n^4+7n^2 \leq n^4+8n^2-5 \quad \text{IFF} \quad 5 \leq n^2 \quad \text{IFF} \quad n \geq 3,$$

SO $\displaystyle\sum_{n=1}^{\infty} \frac{n^2+7}{n^4+8n^2-5}$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.

C) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+n+8}$

COMPARE TO $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$, WHICH DIVERGES SINCE IT'S THE HARMONIC SERIES:

$$\lim_{n\to\infty} \frac{n}{n^2+n+8} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^2}{n^2+n+8} = 1 > 0,$$

SO $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+n+8}$ $\boxed{\text{DIVERGES}}$ BY THE LIMIT COMPARISON TEST.

(OR USE THE COMPARISON TEST WITH $\left(\frac{1}{2}\right)\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$.)

D) $\displaystyle\sum_{n=1}^{\infty} \frac{n+9}{n^3+3n^2-2}$

COMPARE TO $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2}$, WHICH CONVERGES SINCE IT'S A P-SERIES WITH $P>1$:

$$\lim_{n\to\infty} \frac{n+9}{n^3+3n^2-2} \div \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^3+9n^2}{n^3+3n^2-2} = 1 < \infty,$$

SO $\displaystyle\sum_{n=1}^{\infty} \frac{n+9}{n^3+3n^2-2}$ $\boxed{\text{CONVERGES}}$ BY THE LIMIT COMPARISON TEST.

(OR USE THE COMPARISON TEST WITH $(2)\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$.)