MATH 21C — SOLUTIONS - TAN SHEET AND SEC. 10.6

TAN - (15) $\sum_{n=1}^{\infty}(-1)^{n+1}\tan^{-1}\left(\frac{1}{n^2}\right)$ (See Ex. 5, Sec. 10.7)

A) $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\tan^{-1}\left(\frac{1}{n^2}\right)$ COMPARE TO $\sum_{n=1}^{\infty}\frac{1}{n^2}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P>1$:

$$\lim_{n\to\infty}\frac{\tan^{-1}\left(\frac{1}{n^2}\right)}{\frac{1}{n^2}} = \lim_{x\to 0}\frac{\tan^{-1}x}{x} \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to 0}\frac{\frac{1}{1+x^2}}{1} = 1 \neq \infty, \text{ so}$$

(LETTING $x=\frac{1}{n^2}$)

$\sum_{n=1}^{\infty}|a_n|$ CONVERGES BY THE LCT AND $\sum_{n=1}^{\infty}a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

(16) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{\pi}{2n}\right)$

A) $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\sin\left(\frac{\pi}{2n}\right)$ (WHERE $\sin\left(\frac{\pi}{2n}\right)>0$ SINCE $0<\frac{\pi}{2n}\leq\frac{\pi}{2}$ FOR $n\geq 1$)

COMPARE TO $\sum_{n=1}^{\infty}\frac{\pi}{2n}$, WHICH DIVERGES SINCE IT'S A MULTIPLE OF THE HARMONIC SERIES:

$$\lim_{n\to\infty}\frac{\sin\left(\frac{\pi}{2n}\right)}{\frac{\pi}{2n}} = \lim_{x\to 0}\frac{\sin x}{x} = 1 \neq 0, \text{ so } \sum_{n=1}^{\infty}|a_n| \text{ DIVERGES BY THE LCT.}$$

(LETTING $x=\frac{\pi}{2n}$)

B) $\sum_{n=1}^{\infty}(-1)^{n+1}\sin\left(\frac{\pi}{2n}\right)$ CONVERGES BY THE AST SINCE

i) $\lim_{n\to\infty}\sin\left(\frac{\pi}{2n}\right) = \sin 0 = 0$ AND ii) $\sin\left(\frac{\pi}{2n}\right) \geq \sin\left(\frac{\pi}{2(n+1)}\right)$ FOR ALL n

[SINCE $\frac{\pi}{2n} \geq \frac{\pi}{2(n+1)}$ FOR ALL n AND $f(x)=\sin x$ IS INCREASING ON $\left[0,\frac{\pi}{2}\right]$],

SO $\sum_{n=1}^{\infty}a_n$ $\boxed{\text{CONVERGES CONDITIONALLY}}$.

10.6 - (48) $1+\frac{1}{4}-\frac{1}{9}-\frac{1}{16}+\frac{1}{25}+\frac{1}{36}-\frac{1}{49}-\frac{1}{64}+\cdots$ IS $\boxed{\text{ABSOLUTELY CONVERGENT}}$,

SINCE $\sum_{n=1}^{\infty}|a_n| = \sum_{n=1}^{\infty}\frac{1}{n^2}$ CONVERGES BECAUSE IT'S A P-SERIES WITH $P>1$.

(53) $\sum_{n=1}^{\infty}(-1)^n\frac{1}{n^2+3}$ SINCE $|R_n| = |S-S_n| < u_{n+1}$, $|R_n| < .001$ IF

$u_{n+1} = \frac{1}{(n+1)^2+3} < \frac{1}{1000}$:

$\frac{1}{(n+1)^2+3} < \frac{1}{1000}$ IFF $1000 < (n+1)^2+3$ IFF $997 < (n+1)^2$ IFF

$n+1 > \sqrt{997} \approx 31.6$ IFF $n > \sqrt{997}-1 \approx 30.6$, SO $\boxed{n \geq 31}$

WILL GUARANTEE THIS ACCURACY.

MATH 21C SOLUTIONS - Sec. 10.2 AND 10.7

10.2 - (73) $\sum_{n=0}^{\infty} 2^n x^n = \sum_{n=0}^{\infty} (2x)^n$ is a geometric series with $r = 2x$, so it converges iff $-1 < r < 1$ iff $-1 < 2x < 1$ iff $-\frac{1}{2} < x < \frac{1}{2}$; so $\boxed{\left(-\frac{1}{2}, \frac{1}{2}\right)}$ is the interval of convergence. For $-\frac{1}{2} < x < \frac{1}{2}$, the sum $S = \frac{a}{1-r} = \boxed{\frac{1}{1-2x}}$.

(75) $\sum_{n=0}^{\infty} (-1)^n (x+1)^n = \sum_{n=0}^{\infty} \left((-1)(x+1)\right)^n$ is a geometric series with $r = -(x+1)$, so it converges iff $-1 < r < 1$ iff $-1 < -(x+1) < 1$ iff $1 > x+1 > -1$ iff $0 > x > -2$; so $\boxed{(-2, 0)}$ is the interval of convergence.

For $-2 < x < 0$, the sum $S = \frac{a}{1-r} = \frac{1}{1-(-(x+1))} = \boxed{\frac{1}{x+2}}$

10.7 - (9a) $\sum_{n=1}^{\infty} \frac{x^n}{n\sqrt{n}\, 3^n} = \sum_{n=1}^{\infty} \frac{x^n}{n^{3/2} 3^n}$

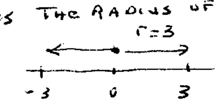
1) $\ell = \lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)^{3/2} 3^{n+1}} \cdot \frac{n^{3/2} 3^n}{|x|^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{3/2} \cdot \frac{|x|}{3}$
$= 1^{3/2} \cdot \frac{|x|}{3} = \frac{|x|}{3}$

2) $\ell < 1$ iff $\frac{|x|}{3} < 1$ iff $|x| < 3$ iff $-3 < x < 3$

3) a) if $x = 3$, we get $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, which converges since it's a p-series, $p > 1$.

b) if $x = -3$, we get $\sum_{n=1}^{\infty} \frac{(-3)^n}{n^{3/2} 3^n} = \sum_{n=1}^{\infty} (-1)^n \cdot \frac{1}{n^{3/2}}$, which converges since $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges (or using the AST).

Thus $\boxed{[-3, 3]}$ is the interval of convergence, and $\boxed{3}$ is the radius of convergence.



(49) a) $\sum_{n=0}^{\infty} (-1)^n \frac{(x-3)^n}{2^n} = 1 - \frac{1}{2}(x-3) + \frac{1}{4}(x-3)^2 - \frac{1}{8}(x-3)^3 + \cdots$ is a geometric series with $r = -\frac{x-3}{2}$, so it converges iff $|r| < 1$:

$\left| -\frac{x-3}{2} \right| < 1$ iff $\frac{|x-3|}{2} < 1$ iff $|x-3| < 2$ iff $1 < x < 5$, so $\boxed{(1, 5)}$ is the interval of convergence.

For $1 < x < 5$, its sum $S = \frac{a}{1-r} = \frac{1}{1 - \left(-\frac{x-3}{2}\right)} = \frac{1}{1 + \frac{x-3}{2}} \cdot \frac{2}{2} = \boxed{\frac{2}{x-1}}$

b) Differentiating this series gives

$\boxed{\sum_{n=1}^{\infty} (-1)^n \frac{n}{2^n} (x-3)^{n-1}} = -\frac{1}{2} + \frac{1}{2}(x-3) - \frac{3}{8}(x-3)^2 + \frac{1}{4}(x-3)^3 - \cdots,$

and its sum is given by $D_x\left(\frac{2}{x-1}\right) = D_x\left(2(x-1)^{-1}\right) = \boxed{-\frac{2}{(x-1)^2}}$

Since differentiation does not change the radius of convergence, the series still converges for $1 < x < 5$.

1) if $x = 5$, we get $\sum_{n=1}^{\infty} (-1)^n \frac{n}{2}$, which diverges by the Divergence Test,

2) if $x = 1$, we get $-\sum_{n=1}^{\infty} \frac{n}{2}$, which also diverges by the Divergence Test,

Therefore this series also has $\boxed{(1, 5)}$ as its interval of convergence.

(10a) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{\sqrt{n}}$

1) $\ell = \lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x-1|^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{|x-1|^n} = \lim_{n\to\infty} \sqrt{\frac{n}{n+1}} \cdot |x-1| = \sqrt{1}\cdot|x-1| = |x-1|$

2) $\ell < 1$ IFF $|x-1| < 1$ IFF $-1 < x-1 < 1$ IFF $\underline{0 < x < 2}$

3) a) IF $\underline{x=0}$, WE HAVE $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$, WHICH CONVERGES BY THE AST SINCE i) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ AND ii) $\frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{n+1}}$ FOR ALL n

b) IF $\underline{x=2}$, WE HAVE $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, WHICH DIVERGES SINCE IT'S A P-SERIES WITH $P \leq 1$.

THUS $\boxed{[0,2)}$ IS THE INTERVAL OF CONV., AND $\boxed{r=1}$ IS THE RADIUS OF CONV.

(14a) $\sum_{n=1}^{\infty} \frac{(x-1)^n}{n^3 3^n}$

1) $\ell = \lim_{n\to\infty} |u_n|^{1/n} = \lim_{n\to\infty} \left(\frac{|x-1|^n}{n^3 3^n}\right)^{1/n} = \lim_{n\to\infty} \frac{|x-1|}{n^{3/n}\cdot 3} = \lim_{n\to\infty} \frac{1}{(n^{1/n})^3} \cdot \frac{|x-1|}{3}$
$= \frac{1}{1^3} \cdot \frac{|x-1|}{3} = \frac{|x-1|}{3}$.

2) $\ell < 1$ IFF $\frac{|x-1|}{3} < 1$ IFF $|x-1| < 3$
IFF $-3 < x-1 < 3$ IFF $\underline{-2 < x < 4}$.

3) a) IF $\underline{x=4}$, WE GET $\sum_{n=1}^{\infty} \frac{1}{n^3}$, WHICH CONVERGES (P-SERIES, $P>1$)

b) IF $\underline{x=-2}$, WE GET $\sum_{n=1}^{\infty} (-1)^n \frac{1}{n^3}$, WHICH CONVERGES SINCE IT'S ABSOLUTELY CONVERGENT (OR USE THE AST).

THUS $\boxed{[-2,4]}$ IS THE INTERVAL OF CONV., AND $\boxed{r=3}$ IS THE RADIUS OF CONV.

(16a) $\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{\sqrt{n}+3}$

1) $\ell = \lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x|^{n+2}}{\sqrt{n+1}+3} \cdot \frac{\sqrt{n}+3}{|x|^{n+1}} = \lim_{n\to\infty} \frac{\sqrt{n}+3}{\sqrt{n+1}+3}\cdot|x|$

$= \lim_{n\to\infty} \frac{\frac{\sqrt{n}}{\sqrt{n}} + \frac{3}{\sqrt{n}}}{\frac{\sqrt{n+1}}{\sqrt{n}} + \frac{3}{\sqrt{n}}} \cdot |x| = \lim_{n\to\infty} \frac{1+\frac{3}{\sqrt{n}}}{\sqrt{\frac{n+1}{n}} + \frac{3}{\sqrt{n}}} \cdot |x| = \frac{1+0}{\sqrt{1}+0}\cdot|x| = |x|$

2) $\ell < 1$ IFF $|x| < 1$ IFF $\underline{-1 < x < 1}$.

3) a) IF $\underline{x=1}$, WE GET $\sum_{n=0}^{\infty} (-1)^n \frac{1}{\sqrt{n}+3}$, WHICH CONVERGES BY THE AST SINCE
i) $\lim_{n\to\infty} \frac{1}{\sqrt{n}+3} = 0$ AND ii) $\frac{1}{\sqrt{n}+3} \geq \frac{1}{\sqrt{n+1}+3}$ FOR ALL n.

b) IF $\underline{x=-1}$, WE GET $\sum_{n=0}^{\infty} \frac{(-1)^{2n+1}}{\sqrt{n}+3} = -\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}+3}$, AND THIS SERIES DIVERGES
SINCE $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}+3}$ DIVERGES: COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$, WHICH DIVERGES SINCE IT'S A P-SERIES WITH $P \leq 1$:

$\lim_{n\to\infty} \frac{1}{\sqrt{n}+3} \div \frac{1}{\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n}+3} = \lim_{n\to\infty} \frac{1}{1+\frac{3}{\sqrt{n}}} = \frac{1}{1+0} = 1 \neq 0$,

SO $\sum_{n=0}^{\infty} \frac{1}{\sqrt{n}+3}$ DIVERGES BY THE LCT.

THUS $\boxed{(-1,1]}$ IS THE INTERVAL OF CONV., AND $\boxed{r=1}$ IS THE RADIUS OF CONV.