SEC. 10.1 – SOLUTIONS

(32) $\displaystyle\lim_{n\to\infty} \frac{n+3}{n^2+5n+6} \div \frac{n^2}{n^2} = \lim_{n\to\infty} \frac{\frac{1}{n}+\frac{3}{n^2}}{1+\frac{5}{n}+\frac{6}{n^2}} = \frac{0}{1} = \boxed{0}$ (CONVERGES)

(OR USE THAT THE LIMIT IS 0 SINCE THE DEGREE ON TOP IS LESS THAN THE DEGREE ON THE BOTTOM.)

(36) $\displaystyle\lim_{n\to\infty} (-1)^n\left(1-\frac{1}{n}\right)$ DOES NOT EXIST SINCE $\displaystyle\lim_{n\to\infty}\left(1-\frac{1}{n}\right)=1$, SO

$a_n \to 1$ FOR n EVEN AND $a_n \to -1$ FOR n ODD. (DIVERGES)

(45) $\displaystyle\lim_{n\to\infty} \frac{\sin n}{n} = \boxed{0}$ BY THE SQUEEZE TH, SINCE $\displaystyle -\frac{1}{n} \le \frac{\sin n}{n} \le \frac{1}{n}$ FOR ALL n

AND $\displaystyle\lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$ AND $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$. (CONVERGES)

(47) $\displaystyle\lim_{n\to\infty} \frac{n}{2^n} = \lim_{x\to\infty}\frac{x}{2^x} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to\infty}\frac{1}{2^x \ln 2} = \boxed{0}$ (CONVERGES)

(49) $\displaystyle\lim_{n\to\infty} \frac{\ln(n+1)}{\sqrt{n}} = \lim_{x\to\infty}\frac{\ln(x+1)}{\sqrt{x}} \underset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to\infty}\frac{\frac{1}{x+1}}{\frac{1}{2\sqrt{x}}} = \lim_{x\to\infty}\frac{2\sqrt{x}}{x+1} \div \frac{x}{x}$

$\displaystyle = \lim_{x\to\infty}\frac{\frac{2}{\sqrt{x}}}{1+\frac{1}{x}} = \frac{0}{1} = \boxed{0}$ (CONVERGES)

(54) $\displaystyle\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{-1}{n}\right)^n = \boxed{e^{-1}} = \boxed{\frac{1}{e}}$ (CONVERGES)

(60) $\displaystyle\lim_{n\to\infty}\left(\ln n - \ln(n+1)\right) = \lim_{n\to\infty}\ln\left(\frac{n}{n+1}\right) = \ln 1 = \boxed{0}$ (CONVERGES)

(SINCE $\displaystyle\frac{n}{n+1} = \frac{1}{1+\frac{1}{n}} \to 1$ AND $f(x)=\ln x$ IS CONTINUOUS AT 1)

(69) $\displaystyle\lim_{n\to\infty}\left(\frac{3n+1}{3n-1}\right)^n = \lim_{n\to\infty}\left(\frac{1+\frac{1}{3n}}{1-\frac{1}{3n}}\right)^n = \lim_{n\to\infty}\frac{\left(1+\frac{1}{3n}\right)^n}{\left(1-\frac{1}{3n}\right)^n}$

$\displaystyle = \lim_{n\to\infty}\frac{\left(1+\frac{1/3}{n}\right)^n}{\left(1+\frac{-1/3}{n}\right)^n} = \frac{e^{1/3}}{e^{-1/3}} = \boxed{e^{2/3}}$ (CONVERGES)

(70) $\displaystyle\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty}\left(\frac{\frac{n}{n}}{\frac{n}{n}+\frac{1}{n}}\right)^n = \lim_{n\to\infty}\left(\frac{1}{1+\frac{1}{n}}\right)^n$

$\displaystyle = \lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n}\right)^n} = \boxed{\frac{1}{e}}$ (CONVERGES)

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(72) $\lim_{n\to\infty} \left(1-\frac{1}{n^2}\right)^n = \lim_{n\to\infty} \left(\left(1+\frac{1}{n}\right)\left(1-\frac{1}{n}\right)\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \left(1+\frac{-1}{n}\right)^n = e^1 \cdot e^{-1} = e^0 = \boxed{1}$

OR $\lim_{n\to\infty} \left(1-\frac{1}{n^2}\right)^n = \lim_{n\to\infty} \left(\left(1-\frac{1}{n^2}\right)^{n^2}\right)^{\frac{1}{n}} = \left(e^{-1}\right)^0 = \boxed{1}$ SINCE

$\lim_{n\to\infty} \left(1-\frac{1}{n^2}\right)^{n^2} = \lim_{m\to\infty} \left(1-\frac{1}{m}\right)^m = \lim_{m\to\infty} \left(1+\frac{-1}{m}\right)^m = e^{-1}$ (CONVERGES)

OR 1) $\lim_{n\to\infty} \ln a_n = \lim_{n\to\infty} \ln\left(1-\frac{1}{n^2}\right)^n = \lim_{n\to\infty} n\ln\left(1-\frac{1}{n^2}\right) = \lim_{x\to\infty} x\ln\left(1-\frac{1}{x^2}\right)$

$= \lim_{x\to\infty} \frac{\ln\left(1-\frac{1}{x^2}\right)}{\frac{1}{x}} \underset{\left(\frac{0}{0}\right)}{=} \lim_{x\to\infty} \frac{\frac{1}{1-\frac{1}{x^2}}\cdot\frac{2}{x^3}}{-\frac{1}{x^2}} \cdot \frac{(-x^2)}{(-x^2)} = \lim_{x\to\infty} \frac{-\frac{2}{x}}{1-\frac{1}{x^2}} = \frac{0}{1} = 0$, so

2) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} e^{\ln a_n} = e^{\lim_{n\to\infty} a_n} = e^0 = \boxed{1}$

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(80) $\lim_{n\to\infty} \left(3^n+5^n\right)^{1/n} = \lim_{n\to\infty} \left(5^n\left(\frac{3^n}{5^n}+1\right)\right)^{1/n} = \lim_{n\to\infty} \left(5^n\right)^{1/n} \left(\left(\frac{3}{5}\right)^n+1\right)^{1/n}$

$= \lim_{n\to\infty} 5\left(\left(\frac{3}{5}\right)^n+1\right)^{1/n} = 5(0+1)^0 = 5\cdot 1^0 = 5\cdot 1 = \boxed{5}$ (CONVERGES)

OR $5^n < 3^n + 5^n < 5^n + 5^n = 2\cdot 5^n$ FOR ALL n, SO

$\left(5^n\right)^{1/n} < \left(3^n+5^n\right)^{1/n} < \left(2\cdot 5^n\right)^{1/n}$ AND THEREFORE

$5 < \left(3^n+5^n\right)^{1/n} < 2^{1/n}\cdot 5$ FOR ALL n.

SINCE $\lim_{n\to\infty} 5 = 5$ AND $\lim_{n\to\infty} 2^{1/n}\cdot 5 = 2^0\cdot 5 = 5$,

$\lim_{n\to\infty} \left(3^n+5^n\right)^{1/n} = \boxed{5}$ BY THE SQUEEZE TH.

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(87) $\lim_{n\to\infty} \left(n-\sqrt{n^2-n}\right) = \lim_{n\to\infty} \left(n-\sqrt{n^2-n}\right)\cdot\left(\frac{n+\sqrt{n^2-n}}{n+\sqrt{n^2-n}}\right) = \lim_{n\to\infty} \frac{n^2-(n^2-n)}{n+\sqrt{n^2-n}}$

$= \lim_{n\to\infty} \frac{n}{n+\sqrt{n^2}\sqrt{1-\frac{1}{n}}} = \lim_{n\to\infty} \frac{n}{n+n\sqrt{1-\frac{1}{n}}} = \lim_{n\to\infty} \frac{n}{n\left(1+\sqrt{1-\frac{1}{n}}\right)}$

$= \lim_{n\to\infty} \frac{1}{1+\sqrt{1-\frac{1}{n}}} = \frac{1}{1+\sqrt{1}} = \boxed{\frac{1}{2}}$

(91) $a_1 = 2$, $a_{n+1} = \frac{72}{1+a_n}$ LET $\lim_{n\to\infty} a_n = L$; THEN $\lim_{n\to\infty} a_{n+1} = L$ ALSO, SO

$\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{72}{1+a_n} \implies L = \frac{72}{1+L} \implies L^2 + L = 72 \implies$

$L^2 + L - 72 = 0 \implies (L+9)(L-8) = 0 \implies L = -9$ OR $L = 8$.

SINCE $a_n > 0$ FOR ALL n, $L \geq 0$; SO $\boxed{L = 8}$