MATH 21C SOLUTIONS – SEC. 10.10 AND 10.9

10.10 – (16) $\displaystyle\int_0^{.4} \frac{e^{-x}-1}{x}\,dx$ $\qquad e^{-x}-1 = \left(1+(-x)+\frac{(-x)^2}{2!}+\frac{(-x)^3}{3!}+\frac{(-x)^4}{4!}-\cdots\right)-1$

$$= \left(\cancel{1} - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right) - \cancel{1}$$

so $\displaystyle\frac{e^{-x}-1}{x} = -1 + \frac{x}{2} - \frac{x^2}{6} + \frac{x^3}{24} - \cdots$ AND

$$\int_0^{.4} \frac{e^{-x}-1}{x}\,dx = \int_0^{.4}\left(-1+\frac{x}{2}-\frac{x^2}{6}+\frac{x^3}{24}-\frac{x^4}{120}+\cdots\right)dx = \left[-x+\frac{x^2}{4}-\frac{x^3}{18}+\frac{x^4}{96}-\frac{x^5}{600}+\cdots\right]_0^{.4}$$

$$= -(.4) + \frac{(.4)^2}{4} - \frac{(.4)^3}{18} + \frac{(.4)^4}{96} - \frac{(.4)^5}{600} + \cdots$$

$$\approx \boxed{-(.4) + \frac{(.4)^2}{4} - \frac{(.4)^3}{18} + \frac{(.4)^4}{96} - \frac{(.4)^5}{600}} \quad \text{WITH } \underline{|E| < \frac{(.4)^6}{6(720)}} < 10^{-5}$$

(21) $\displaystyle\int_0^{.1}\sqrt{1+x^4}\,dx$ $\qquad \sqrt{1+x} = (1+x)^{1/2} = 1+\frac{1}{2}x+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3+\frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4+\cdots$

so $\displaystyle\sqrt{1+x^4} = 1+\frac{1}{2}x^4-\frac{1}{8}x^8+\frac{1}{16}x^{12}-\frac{5}{128}x^{16}+\cdots$ AND

$$\int_0^{.1}\sqrt{1+x^4}\,dx = \int_0^{.1}\left(1+\frac{1}{2}x^4-\frac{1}{8}x^8+\cdots\right)dx = \left[x+\frac{x^5}{10}-\frac{x^9}{72}+\frac{1}{208}x^{13}-\cdots\right]_0^{.1}$$

$$= .1 + \frac{(.1)^5}{10} - \frac{(.1)^9}{72} + \frac{1}{208}(.1)^{13} - \cdots \approx \boxed{.1 + \frac{(.1)^5}{10}} \quad \text{WITH } \underline{|E| < \frac{(.1)^9}{72}} < 10^{-8}$$

(NOTICE THAT THE SERIES IS ALTERNATING IN SIGN, STARTING WITH THE 2ND TERM.)

(32) $$\lim_{\theta\to 0}\frac{\sin\theta-\theta+\frac{\theta^3}{6}}{\theta^5} = \lim_{\theta\to 0}\frac{\left[\cancel{\theta}-\cancel{\frac{\theta^3}{3!}}+\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots\right]-\cancel{\theta}+\cancel{\frac{\theta^3}{6}}}{\theta^5}$$

$$= \lim_{\theta\to 0}\frac{\frac{\theta^5}{5!}-\frac{\theta^7}{7!}+\cdots}{\theta^5} = \lim_{\theta\to 0}\left[\frac{1}{5!}-\frac{\theta^2}{7!}+\cdots\right] = \frac{1}{5!} = \boxed{\frac{1}{120}}$$

(34) $$\lim_{y\to 0}\frac{\tan^{-1}y-\sin y}{y^3\cos y} = \lim_{y\to 0}\frac{\left(\cancel{y}-\frac{y^3}{3}+\frac{y^5}{5}-\cdots\right)-\left(\cancel{y}-\frac{y^3}{6}+\frac{y^5}{120}-\cdots\right)}{y^3\left(1-\frac{y^2}{2}+\frac{y^4}{24}+\cdots\right)}$$

$$= \lim_{y\to 0}\frac{-\frac{1}{6}y^3+\frac{23}{120}y^5-\cdots}{y^3\left(1-\frac{y^2}{2}+\frac{y^4}{24}-\cdots\right)} = \lim_{y\to 0}\frac{\cancel{y^3}\left(-\frac{1}{6}+\frac{23}{120}y^2+\cdots\right)}{\cancel{y^3}\left(1-\frac{y^2}{2}+\cdots\right)} = \frac{-\frac{1}{6}}{1} = \boxed{-\frac{1}{6}}$$

(54) $\displaystyle\ln(1+x) = x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots$, so

$$\ln 1.1 \approx .1 - \frac{(.1)^2}{2} + \frac{(.1)^3}{3} - \frac{(.1)^4}{4} + \frac{(.1)^5}{5} - \frac{(.1)^6}{6} + \frac{(.1)^7}{7} \quad \text{WITH } |E| < \frac{(.1)^8}{8} < 10^{-8},$$

so <u>THE FIRST 7 NONZERO TERMS</u> GUARANTEE THIS ACCURACY.

10.9 – (38) $\displaystyle\cos x = 1-\frac{x^2}{2!}+\frac{x^4}{4!}-\cdots \approx 1-\frac{x^2}{2!}$,

WITH $\underline{|E| < \frac{|x|^4}{4!} < \frac{(.5)^4}{24} = \frac{1}{384}}$ FOR $|x|<.5$ USING THE AST.

THIS ESTIMATE IS <u>TOO SMALL</u>, SINCE THE FIRST TERM OMITTED IS POSITIVE.

MATH 21C   PINK SHEET - SOLUTIONS

(5) $f(x) = \sqrt{x+3}, \quad a = 1$

$$f(x) = (x+3)^{1/2}$$
$$f'(x) = \tfrac{1}{2}(x+3)^{-1/2}$$
$$f''(x) = \tfrac{1}{2}\left(-\tfrac{1}{2}\right)(x+3)^{-3/2}$$
$$f'''(x) = \tfrac{1}{2}\left(-\tfrac{1}{2}\right)\left(-\tfrac{3}{2}\right)(x+3)^{-5/2}$$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	2	2
1	$\frac{1}{2}\cdot\frac{1}{2}$	$1/4$
2	$-\frac{1}{4}\cdot\frac{1}{2^3}$	$-1/64$
3	$\frac{3}{8}\cdot\frac{1}{2^5}$	$\frac{1}{512}$

$$\sqrt{x+3} = \boxed{2 + \frac{1}{4}(x-1) - \frac{1}{64}(x-1)^2 + \frac{1}{512}(x-1)^3 - \cdots}$$

$$= \boxed{2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-3)}{2^{3n-1}\, n!}(x-1)^n}$$

(USING THE CONVENTION THAT AN "EMPTY PRODUCT" EQUALS 1)

$$= \boxed{2 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-2)!}{2^{4n-2}\, n!\,(n-1)!}(x-1)^n}$$

(MULTIPLYING BY $2\cdot 4\cdot 6\cdots(2n-2) = 2^{n-1}(n-1)!$ ON THE TOP AND BOTTOM)

[OR use $\sqrt{x+3} = \sqrt{4+(x-1)} = \sqrt{4}\sqrt{1+\frac{x-1}{4}} = 2\sqrt{1+u} = 2(1+u)^{1/2}$ WHERE $u = \frac{x-1}{4}$]

(6)

$$
\begin{array}{r|l}
 & x + \frac{x^3}{3} + \frac{2}{15}x^5 + \cdots \\
\hline
1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots & x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \\
 & x - \frac{x^3}{2} + \frac{x^5}{24} - \cdots \\
\hline
 & \quad \frac{x^3}{3} - \frac{x^5}{30} + \cdots \\
 & \quad \frac{x^3}{3} - \frac{x^5}{6} + \cdots \\
\hline
 & \qquad \frac{2}{15}x^5 - \cdots
\end{array}
$$

← SINCE $\tan x = \dfrac{\sin x}{\cos x}$

SO $\boxed{\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}$ (see 10.7, #57)

(7) A) THE MACLAURIN SERIES FOR $\ln(1+x)$ HAS INTERVAL OF CONVERGENCE $(-1,1]$, SO LETTING $x=2$ WOULD GIVE A DIVERGENT SERIES.

(SINCE $\ln\frac{1}{3} = -\ln 3$, WE COULD USE $\ln 3 = -\ln\frac{1}{3} = -\ln\left(1+\left(-\frac{2}{3}\right)\right)$ TO FIND AN APPROXIMATION FOR $\ln 3$, BUT THE APPROXIMATION IN PART B) IS MUCH MORE ACCURATE.)

B) BY #53 IN SEC. 10.10,*

$$\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right) \text{ FOR } |x|<1,$$

LETTING $\frac{1+x}{1-x} = 3$ GIVES $1+x = 3-3x$, SO $4x=2$ AND $x=\frac{1}{2}$;

$$\text{SO } \ln 3 = 2\left(\frac{1}{2} + \frac{\left(\frac{1}{2}\right)^3}{3} + \frac{\left(\frac{1}{2}\right)^5}{5} + \frac{\left(\frac{1}{2}\right)^7}{7} + \cdots\right) \approx \boxed{1 + \frac{1}{12} + \frac{1}{80} + \frac{1}{448}}.$$

(8) LET $\sum_{n=1}^{\infty} a_n$ BE A CONDITIONALLY CONVERGENT SERIES,

SO $\sum_{n=1}^{\infty} |a_n|$ DIVERGES BUT $\sum_{n=1}^{\infty} a_n$ CONVERGES.

IF WE LET $b_n = a_n^+ = \frac{1}{2}(|a_n| + a_n) = \begin{cases} a_n, & \text{IF } a_n \geq 0 \\ 0, & \text{IF } a_n < 0 \end{cases}$

AND $c_n = a_n^- = \frac{1}{2}(|a_n| - a_n) = \begin{cases} 0, & \text{IF } a_n \geq 0 \\ -a_n, & \text{IF } a_n < 0, \end{cases}$

THEN $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (b_n - c_n)$ WHERE THE SERIES

$\sum_{n=1}^{\infty} b_n = \frac{1}{2}\sum_{n=1}^{\infty} (|a_n| + a_n)$ AND $\sum_{n=1}^{\infty} c_n = \frac{1}{2}\sum_{n=1}^{\infty} (|a_n| - a_n)$

ARE NONNEGATIVE-TERM SERIES WHICH BOTH DIVERGE

SINCE $\sum_{n=1}^{\infty} |a_n|$ DIVERGES AND $\sum_{n=1}^{\infty} a_n$ CONVERGES.

REMARK THIS IS THE KEY IDEA USED IN THE PROOF THAT A CONDITIONALLY CONVERGENT SERIES CAN BE REARRANGED SO THAT IT CONVERGES TO ANY SPECIFIED NUMBER, OR SO THAT IT DIVERGES.

10.10 - (53)* $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \frac{x^6}{6} + \frac{x^7}{7} - \cdots$ IF $|x|<1$, SO

$\ln(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \frac{x^6}{6} - \frac{x^7}{7} - \cdots$ IF $|x|<1$

(SINCE $(-x)^n = (-1)^n x^n = \begin{cases} x^n, & \text{IF } n \text{ IS EVEN} \\ -x^n, & \text{IF } n \text{ IS ODD} \end{cases}$)

SUBTRACTING THESE EQUATIONS GIVES

$$\ln(1+x) - \ln(1-x) = 2x + 2\cdot\frac{x^3}{3} + 2\cdot\frac{x^5}{5} + 2\cdot\frac{x^7}{7} + \cdots,$$

$$\text{SO } \ln\left(\frac{1+x}{1-x}\right) = 2\left[x + \frac{x^3}{3} + \frac{x^5}{5} + \frac{x^7}{7} + \cdots\right] \text{ IF } |x|<1$$

10.9 – (39) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ SINCE THIS IS AN ALTERNATING SERIES FOR $x \neq 0$

AND THE TERMS ARE DECREASING IN ABSOLUTE VALUE WHEN $|x| < 10^{-3}$,

$\sin x \approx x$ WITH $\underline{|E| < \frac{|x|^3}{3!} < \frac{(10^{-3})^3}{6} = \frac{1}{6(10^9)}}$.

$\sin x > x$ WHEN THE FIRST TERM OMITTED IS POSITIVE, AND $-\frac{x^3}{6} > 0$ GIVES

$x^3 < 0$ SO $x < 0$: $\underline{-.001 < x < 0}$.

(40) $\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}(-\frac{1}{2})}{2!}x^2 + \frac{(\frac{1}{2})(-\frac{1}{2})(-\frac{3}{2})}{3!}x^3 + \cdots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$

a) IF $\underline{0 < x < .01}$, WE GET $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ WITH $\underline{|E| < \frac{x^2}{8} < \frac{(.01)^2}{8} = \frac{1}{80{,}000}}$

USING THE AST.

b) IF $\underline{-.01 < x < 0}$, WE GET $\sqrt{1+x} \approx 1 + \frac{1}{2}x$ WITH $\underline{E = R_1(x) = \frac{f^{(2)}(c)}{2!}(x-0)^2}$,

SO $|E| = \left| \frac{-\frac{1}{4}(1+c)^{-3/2}}{2!} x^2 \right| = \frac{|x|^2}{8(1+c)^{3/2}}$ WHERE $x < c < 0$ AND THEREFORE

$\underline{|E| < \frac{(.01)^2}{8(1-.01)^{3/2}} = \frac{1}{80{,}000\,(.99)^{3/2}}}$ SINCE $(1+t)^{-3/2}$ IS DECREASING ON $[-.01, 0]$.

(41) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ WITH $|x| < .1$, SO

$e^x \approx 1 + x + \frac{x^2}{2}$ WITH $|E| = |R_2(x)| = \left| \frac{f^{(3)}(c)}{3!}(x-0)^3 \right| = \frac{e^c}{6}|x|^3$ WHERE c IS BETWEEN 0 AND x,

SO $c < .1$ AND $\underline{|E| < \frac{e^{.1}}{6}(.1)^3 < \frac{3^{.1}}{6000}}$ (USING THAT $e < 3$).

[FOR $\underline{-.1 < x < 0}$, WE CAN GET THE ESTIMATE $|E| < \frac{1}{6000}$ USING THE AST.*]

(43) a) $\underline{\sin^2 x} = \frac{1}{2}(1 - \cos 2x) = \frac{1}{2}\left(1 - \left[1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \frac{(2x)^8}{8!} - \cdots\right]\right)$

$= \frac{1}{2}\left(\frac{4x^2}{2!} - \frac{16x^4}{4!} + \frac{64x^6}{6!} - \frac{256x^8}{8!} + \cdots\right) = \underline{\frac{2x^2}{2!} - \frac{8x^4}{4!} + \frac{32x^6}{6!} - \frac{128x^8}{8!} + \cdots}$

b) $\underline{2\sin x \cos x} = D_x(\sin^2 x) = \frac{4x}{2!} - \frac{32x^3}{4!} + \frac{192x^5}{6!} - \frac{1024x^7}{8!} + \cdots$

$= \underline{2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots}$

c) $\underline{\sin 2x} = (2x) - \frac{(2x)^3}{3!} + \frac{(2x)^5}{5!} - \frac{(2x)^7}{7!} + \cdots = \underline{2x - \frac{8x^3}{3!} + \frac{32x^5}{5!} - \frac{128x^7}{7!} + \cdots}$

10.10 – (37) $\lim_{x \to 0} \frac{\ln(1+x^2)}{1-\cos x} = \lim_{x \to 0} \frac{x^2 - \frac{(x^2)^2}{2} + \frac{(x^2)^3}{3} - \cdots}{1 - \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)}$

$= \lim_{x \to 0} \frac{x^2 - \frac{x^4}{2} + \frac{x^6}{3} - \cdots}{\frac{x^2}{2} - \frac{x^4}{24} + \frac{x^6}{720} - \cdots} = \lim_{x \to 0} \frac{\cancel{x^2}\left(1 - \frac{x^2}{2} + \frac{x^4}{3} - \cdots\right)}{\cancel{x^2}\left(\frac{1}{2} - \frac{x^2}{24} + \frac{x^4}{720} - \cdots\right)}$

$= \lim_{x \to 0} \frac{1 - \frac{x^2}{2} + \frac{x^4}{3} - \cdots}{\frac{1}{2} - \frac{x^2}{24} + \frac{x^4}{720} - \cdots} = \frac{1}{1/2} = \boxed{2}$

* (OR USING THAT $e^c < e^0 = 1$ IF $c < 0$)