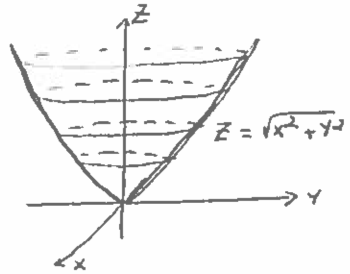
MATH 21C SOLUTIONS – 14.1 – 14.3

14.1 – 40a) $f(x,y) = \sqrt{x^2+y^2}$



TRACE IN THE YZ-PLANE:

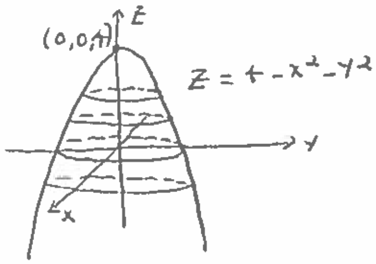
$x = 0$ GIVES $z = \sqrt{y^2} = |y|$

TRACE IN THE PLANE $Z = K$:

$\sqrt{x^2+y^2} = k$, SO $x^2+y^2 = k^2$ (CIRCLE, FOR $k>0$)

(THE GRAPH IS THE TOP HALF OF THE CIRCULAR CONE $z^2 = x^2+y^2$)

42a) $f(x,y) = 4 - x^2 - y^2$



TRACE IN THE YZ-PLANE:

$x = 0$ GIVES $z = 4 - y^2$

TRACE IN THE PLANE $Z = K$:

$k = 4 - x^2 - y^2$ GIVES $x^2 + y^2 = 4 - k$ (CIRCLE, FOR $k < 4$)

(THE GRAPH IS A CIRCULAR PARABOLOID.)

14.2 – 32) a) $f(x,y) = \dfrac{x+y}{x-y}$ IS CONTINUOUS AT $\boxed{\text{ALL POINTS } (x,y) \text{ WHICH ARE NOT ON THE LINE } y = x.}$

b) $f(x,y) = \dfrac{y}{x^2+1}$ IS CONTINUOUS AT $\boxed{\text{ALL POINTS IN THE XY-PLANE}}$.

(SINCE x^2+1 IS NEVER 0)

61) $$\lim_{(x,y)\to(0,0)} \frac{x^3 - xy^2}{x^2+y^2} = \lim_{r\to 0} \frac{r^3\cos^3\theta - (r\cos\theta)(r^2\sin^2\theta)}{r^2}$$

$$= \lim_{r\to 0} \frac{r^3(\cos^3\theta - \cos\theta\sin^2\theta)}{r^2} = \lim_{r\to 0} r\left(\cos^3\theta - \cos\theta\sin^2\theta\right) = \boxed{0}$$

(SINCE $|\cos^3\theta - \cos\theta\sin^2\theta| = |\cos\theta|\,|\cos^2\theta - \sin^2\theta| = |\cos\theta|\,|\cos 2\theta| \le 1$).

14.3 – 12) $f(x,y) = \tan^{-1}\left(\frac{y}{x}\right)$

$$f_x = \boxed{\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(-\frac{y}{x^2}\right)} = \boxed{-\frac{y}{x^2+y^2}}$$

$$f_y = \boxed{\frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\frac{1}{x}} = \boxed{\frac{x}{x^2+y^2}}$$

14) $f(x,y) = e^{-x}\sin(x+y)$

$$f_x = \boxed{e^{-x}\cos(x+y)\cdot 1 + e^{-x}(-1)\sin(x+y)}$$

$$f_y = \boxed{e^{-x}\cos(x+y)\cdot 1}$$

18) $f(x,y) = e^{xy}\ln y$

$$f_x = \boxed{(e^{xy}\cdot y)\ln y}$$

$$f_y = \boxed{e^{xy}\cdot\frac{1}{y} + (e^{xy}\cdot x)\ln y}$$

(19) $f(x,y) = x^y$ $\quad \underline{f_x = y x^{y-1}}$ (POWER RULE), $\quad \underline{f_y = x^y \ln x}$ (FOR $x>0$) (SEE P. 181, (4))

(21) $f(x,y) = \int_x^y g(t)\,dt$

$$\underline{f_x} = \frac{\partial}{\partial x}\left(\int_x^y g(t)\,dt\right) = \frac{\partial}{\partial x}\left(-\int_y^x g(t)\,dt\right) = \underline{-g(x)}$$

(USING PART I OF THE FUND. TH. OF CALCULUS)

$$\underline{f_y} = \frac{\partial}{\partial y}\left(\int_x^y g(t)\,dt\right) = \underline{g(y)}$$

(48) $w = y e^{x^2-y}$

a) $\underline{w_x} = y\left(e^{x^2-y}\cdot 2x\right)$ so

1) $\underline{w_{xx}} = 2y\left(e^{x^2-y}\cdot 1 + \left(e^{x^2-y}\cdot 2x\right)\cdot x\right) = \boxed{2y e^{x^2-y}(1+2x^2)}$

2) $\underline{w_{xy}} = 2x\left(y\left(e^{x^2-y}(-1)\right) + 1\cdot e^{x^2-y}\right) = \boxed{2x e^{x^2-y}(1-y)}$

b) $\underline{w_y} = y\left(e^{x^2-y}(-1)\right) + 1\cdot e^{x^2-y} = e^{x^2-y}(1-y)$

so 3) $\underline{w_{yx}} = (1-y)\left(e^{x^2-y}\cdot 2x\right) = \boxed{2x(1-y)e^{x^2-y}}$

4) $\underline{w_{yy}} = e^{x^2-y}(-1) + \left(e^{x^2-y}(-1)\right)(1-y) = \boxed{e^{x^2-y}(y-2)}$

(60) $f(x,y) = \dfrac{\sin(x^3+y^4)}{x^2+y^2}$ IF $(x,y) \neq (0,0)$, AND $f(0,0)=0$.

a) $$\underline{f_x(0,0)} = \lim_{h\to 0} \frac{f(0+h,0)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin h^3}{h^2} - 0}{h}$$
$$= \lim_{h\to 0} \frac{\sin h^3}{h^3} = \lim_{u\to 0} \frac{\sin u}{u} = \boxed{1}$$ (LETTING $u = h^3$)

b) $$\underline{f_y(0,0)} = \lim_{h\to 0} \frac{f(0,0+h)-f(0,0)}{h} = \lim_{h\to 0} \frac{\frac{\sin h^4}{h^2} - 0}{h} = \lim_{h\to 0} \frac{\sin h^4}{h^3}$$
$$= \lim_{h\to 0}\left(h\cdot\frac{\sin h^4}{h^4}\right) = \left(\lim_{h\to 0} h\right)\left(\lim_{u\to 0}\frac{\sin u}{u}\right) = 0\cdot 1 = \boxed{0}$$ (USING $u = h^4$)

(64) a) $m = \underline{f_y(-1,1)} = 3y^2\big|_{(-1,1)} = \boxed{3}$ $\quad$ b) $m = \underline{f_x(-1,1)} = 2x\big|_{(-1,1)} = \boxed{-2}$

(66) $xz + y\ln x - x^2 + 4 = 0$; FIND $\dfrac{\partial x}{\partial z}$ AT $(1,-1,-3)$

DIFFERENTIATING WITH RESPECT TO z, TREATING y AS A CONSTANT, GIVES

$$x\cdot 1 + \frac{\partial x}{\partial z}\cdot z + y\left(\frac{1}{x}\cdot\frac{\partial x}{\partial z}\right) - 2x\cdot\frac{\partial x}{\partial z} = 0.$$ LETTING $x=1,\ y=-1,\ z=-3$ GIVES

$$1 - 3\cdot\frac{\partial x}{\partial z} + \frac{-1}{1}\cdot\frac{\partial x}{\partial z} - 2\cdot\frac{\partial x}{\partial z} = 0$$ so $1 = 6\cdot\dfrac{\partial x}{\partial z}$ AND $\dfrac{\partial x}{\partial z} = \boxed{\dfrac{1}{6}}$

OR LET $F(x,y,z) = xz + y\ln x - x^2 + 4$; AS ON P. 827,

$$\frac{\partial x}{\partial z} = -\frac{F_z}{F_x} = -\frac{x}{z + \frac{y}{x} - 2x}$$ so $$\left.\frac{\partial x}{\partial z}\right|_{(1,-1,-3)} = -\frac{1}{-3-1-2} = \boxed{\frac{1}{6}}$$

(25) $x^3 - 2y^2 + xy = 0$, $(1,1)$ IF $F(x,y) = x^3 - 2y^2 + xy$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{3x^2+y}{-4y+x}, \quad \text{so} \quad \left.\frac{dy}{dx}\right|_{(1,1)} = -\frac{4}{-3} = \boxed{\frac{4}{3}}$$

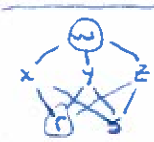
(27) $x^2 + xy + y^2 - 7 = 0$, $(1,2)$ IF $F(x,y) = x^2 + xy + y^2 - 7$,

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2x+y}{x+2y}, \quad \text{so} \quad \left.\frac{dy}{dx}\right|_{(1,2)} = \boxed{-\frac{4}{5}}$$

(33) $w = (x+y+z)^2$, $x = r-s$, $y = \cos(r+s)$, $z = \sin(r+s)$

FIND $\frac{\partial w}{\partial r}$ WHEN $r = 1$, $s = -1$.

A) WITH THE CHAIN RULE:



$$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} + \frac{\partial w}{\partial z}\cdot\frac{\partial z}{\partial r}$$

$$= 2(x+y+z)\cdot 1 + 2(x+y+z)(-\sin(r+s)) + 2(x+y+z)(\cos(r+s))$$

WHEN $r = 1$ AND $s = -1$, $x = 2$, $y = \cos 0 = 1$, AND $z = \sin 0 = 0$,

$$\text{so} \quad \left.\frac{\partial w}{\partial r}\right|_{(1,-1)} = 2\cdot 3\cdot 1 + 2\cdot 3\cdot 0 + 2\cdot 3\cdot 1 = \boxed{12}$$

B) WITHOUT THE CHAIN RULE:

$$w = (x+y+z)^2 = (r - s + \cos(r+s) + \sin(r+s))^2,$$

$$\text{so} \quad \frac{\partial w}{\partial r} = 2(r - s + \cos(r+s) + \sin(r+s))\left[1 - \sin(r+s) + \cos(r+s)\right]$$

$$\text{AND} \quad \left.\frac{\partial w}{\partial r}\right|_{(1,-1)} = 2(2+1+0)\left[1-0+1\right] = 2\cdot 3\cdot 2 = \boxed{12}$$

(35) $w = x^2 + \frac{y}{x}$, $x = u - 2v + 1$, $y = 2u + v - 2$

FIND $\frac{\partial w}{\partial v}$ WHEN $u = 0$, $v = 0$

A) WITH THE CHAIN RULE:

$$\frac{\partial w}{\partial v} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial v} = \left(2x - \frac{y}{x^2}\right)(-2) + \left(\frac{1}{x}\right)(1)$$

WHEN $u = 0$ AND $v = 0$, $x = 1$ AND $y = -2$

$$\text{so} \quad \frac{\partial w}{\partial v} = (2-(-2))(-2) + 1 = \boxed{-7}$$

B) WITHOUT THE CHAIN RULE:

$$w = x^2 + \frac{y}{x} = (u - 2v + 1)^2 + \frac{2u + v - 2}{u - 2v + 1}, \quad \text{so}$$

$$\frac{\partial w}{\partial v} = 2(u - 2v + 1)(-2) + \frac{(u - 2v + 1)\cdot 1 - (2u + v - 2)(-2)}{(u - 2v + 1)^2} \quad \text{AND } u = 0,\ v = 0 \text{ GIVES}$$

$$\left.\frac{\partial w}{\partial v}\right|_{(0,0)} = 2(1)(-2) + \frac{1 - (-2)(-2)}{1^2} = -4 + (-3) = \boxed{-7}$$