MATH 21C　　　SOLUTIONS — SEC. 14.4

(39) $z^3 - xy + yz + y^3 - 2 = 0$; $(1,1,1)$　　LET $F(x,y,z) = z^3 - xy + yz + y^3 - 2$

(a) $$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{-y}{3z^2+y}, \quad \text{so} \quad \left.\frac{\partial z}{\partial x}\right|_{(1,1,1)} = \boxed{\frac{1}{4}}$$

OR　TAKING THE PARTIAL DERIVATIVE OF BOTH SIDES WITH RESPECT TO X GIVES

$$3z^2 \cdot \frac{\partial z}{\partial x} - y + y \cdot \frac{\partial z}{\partial x} = 0 ; \quad \text{so AT } (1,1,1),$$

$$3 \cdot \frac{\partial z}{\partial x} - 1 + \frac{\partial z}{\partial x} = 0 \quad \text{AND} \quad \frac{\partial z}{\partial x} = \boxed{\frac{1}{4}}$$

(b) $$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-x + z + 3y^2}{3z^2+y}, \quad \text{so} \quad \left.\frac{\partial z}{\partial y}\right|_{(1,1,1)} = \boxed{-\frac{3}{4}}$$

OR　TAKING THE PARTIAL DERIVATIVE OF BOTH SIDES WITH RESPECT TO Y GIVES

$$3z^2 \cdot \frac{\partial z}{\partial y} - x + y \cdot \frac{\partial z}{\partial y} + 1 \cdot z + 3y^2 = 0 ; \quad \text{so AT } (1,1,1),$$

$$3 \cdot \frac{\partial z}{\partial y} - 1 + \frac{\partial z}{\partial y} + 1 + 3 = 0 \quad \text{AND} \quad \frac{\partial z}{\partial y} = \boxed{-\frac{3}{4}}$$

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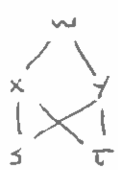
(40) $w = f\left(ts^2, \frac{s}{t}\right)$, $\frac{\partial f}{\partial x}(x,y) = xy$, $\frac{\partial f}{\partial y}(x,y) = \frac{x^2}{2}$

LET $x = ts^2$ AND $y = \frac{s}{t}$:



a) $$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial t} = f_x \cdot \frac{\partial x}{\partial t} + f_y \cdot \frac{\partial y}{\partial t}$$

$$= (xy) \cdot s^2 + \frac{x^2}{2} \cdot \left(-\frac{s}{t^2}\right) = s^3 \cdot s^2 + \frac{1}{2} t^2 s^4 \left(-\frac{s}{t^2}\right)$$

$$= s^5 - \frac{1}{2} s^5 = \boxed{\frac{1}{2} s^5}$$

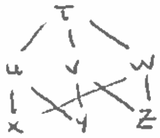
b) $$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} = f_x \cdot \frac{\partial x}{\partial s} + f_y \cdot \frac{\partial y}{\partial s}$$

$$= (xy)(2ts) + \frac{x^2}{2}\left(\frac{1}{t}\right) = s^3(2ts) + \frac{1}{2} t^2 s^4 \left(\frac{1}{t}\right)$$

$$= 2ts^4 + \frac{1}{2} ts^4 = \boxed{\frac{5}{2} ts^4}$$

MATH 21C SOLUTIONS – Sec. 14.4

(43) Let $T = f(u, v, w)$ where $u = x - y$, $v = y - z$, $w = z - x$



$$\frac{\partial T}{\partial x} = \frac{\partial T}{\partial u}\cdot\frac{\partial u}{\partial x} + \frac{\partial T}{\partial w}\cdot\frac{\partial w}{\partial x} = f_u(1) + f_w(-1) = f_u - f_w$$

$$\frac{\partial T}{\partial y} = \frac{\partial T}{\partial u}\cdot\frac{\partial u}{\partial y} + \frac{\partial T}{\partial v}\cdot\frac{\partial v}{\partial y} = f_u(-1) + f_v(1) = f_v - f_u$$

$$\frac{\partial T}{\partial z} = \frac{\partial T}{\partial v}\cdot\frac{\partial v}{\partial z} + \frac{\partial T}{\partial w}\cdot\frac{\partial w}{\partial z} = f_v(-1) + f_w(1) = f_w - f_v$$

Thus $\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y} + \frac{\partial T}{\partial z} = (f_u - f_w) + (f_v - f_u) + (f_w - f_v) = 0.$

(49a) $T = f(x, y)$ where $x = \cos t$, $y = \sin t$, $0 \le t \le 2\pi$

1) $\underline{\frac{dT}{dt}} = \frac{\partial T}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial T}{\partial y}\cdot\frac{dy}{dt} = (8x - 4y)(-\sin t) + (8y - 4x)(\cos t)$

$= (8\cos t - 4\sin t)(-\sin t) + (8\sin t - 4\cos t)(\cos t)$

$= 4\sin^2 t - 4\cos^2 t = -4(\cos^2 t - \sin^2 t) = \underline{-4\cos 2t}$

2) So $\underline{\frac{dT}{dt} = 0}$ iff $\cos 2t = 0$ iff $2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}$ (since $0 \le 2t \le 4\pi$)

iff $\underline{t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}}$

3) Since $\frac{d^2T}{dt^2} = -4(-2\sin 2t) = 8\sin 2t$,

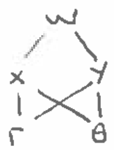
$\underline{\frac{d^2T}{dt^2} > 0}$ for $t = \frac{\pi}{4}, \frac{5\pi}{4}$ and $\underline{\frac{d^2T}{dt^2} < 0}$ for $t = \frac{3\pi}{4}, \frac{7\pi}{4}$;

So the <u>min.</u> temp. occurs at $\underline{\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ and } \left(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)}$ (when $t = \frac{\pi}{4}, \frac{5\pi}{4}$)

and the <u>max.</u> temp. occurs at $\underline{\left(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right) \text{ and } \left(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}\right)}$ (when $t = \frac{3\pi}{4}, \frac{7\pi}{4}$)

SOLUTIONS — Sec. 14.4

(44) $w = f(x,y)$, $x = r\cos\theta$, $y = r\sin\theta$

a) 

1) $$\frac{\partial w}{\partial r} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial r} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial r} = \underline{f_x\cos\theta + f_y\sin\theta} \qquad \text{(EQ. 1)}$$

2) $$\frac{\partial w}{\partial \theta} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial \theta} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial \theta} = f_x(-r\sin\theta) + f_y(r\cos\theta),$$

SO DIVIDING BY r GIVES $$\underline{\frac{1}{r}\cdot\frac{\partial w}{\partial \theta} = -f_x\sin\theta + f_y\cos\theta} \qquad \text{(EQ. 2)}$$

b) 1) MULT. BY $\cos\theta$ IN EQ. 1 AND BY $\sin\theta$ IN EQ. 2 AND THEN SUBTRACT:

$$\begin{aligned} &\frac{\partial w}{\partial r}\cos\theta = f_x\cos^2\theta + f_y\sin\theta\cos\theta \\ - \quad &\frac{1}{r}\cdot\frac{\partial w}{\partial \theta}\sin\theta = -f_x\sin^2\theta + f_y\cos\theta\sin\theta \\ \hline &\frac{\partial w}{\partial r}\cos\theta - \frac{1}{r}\cdot\frac{\partial w}{\partial \theta}\sin\theta = f_x\cos^2\theta + f_x\sin^2\theta = f_x(\cos^2\theta + \sin^2\theta) = f_x\cdot 1, \end{aligned}$$

SO $$\boxed{f_x = \frac{\partial w}{\partial r}\cos\theta - \frac{1}{r}\cdot\frac{\partial w}{\partial \theta}\sin\theta}$$

2) MULT. BY $\sin\theta$ IN EQ. 1 AND BY $\cos\theta$ IN EQ. 2 AND THEN ADD:

$$\begin{aligned} &\frac{\partial w}{\partial r}\sin\theta = f_x\cos\theta\sin\theta + f_y\sin^2\theta \\ + \quad &\frac{1}{r}\cdot\frac{\partial w}{\partial \theta}\cos\theta = -f_x\sin\theta\cos\theta + f_y\cos^2\theta \\ \hline &\frac{\partial w}{\partial r}\sin\theta + \frac{1}{r}\cdot\frac{\partial w}{\partial \theta}\cos\theta = f_y\sin^2\theta + f_y\cos^2\theta = f_y(\sin^2\theta + \cos^2\theta) = f_y\cdot 1, \end{aligned}$$

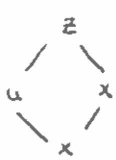
SO $$\boxed{f_y = \frac{\partial w}{\partial r}\sin\theta + \frac{1}{r}\cdot\frac{\partial w}{\partial \theta}\cos\theta}$$

c) $$\underline{(f_x)^2 + (f_y)^2} = \left(w_r\cos\theta - \frac{1}{r}w_\theta\sin\theta\right)^2 + \left(w_r\sin\theta + \frac{1}{r}w_\theta\cos\theta\right)^2$$

$$= \left(w_r^2\cos^2\theta \cancel{- \frac{2}{r}w_r w_\theta\cos\theta\sin\theta} + \frac{1}{r^2}w_\theta^2\sin^2\theta\right) +$$
$$\left(w_r^2\sin^2\theta \cancel{+ \frac{2}{r}w_r w_\theta\sin\theta\cos\theta} + \frac{1}{r^2}w_\theta^2\cos^2\theta\right)$$
$$= w_r^2(\cos^2\theta + \sin^2\theta) + \frac{1}{r^2}w_\theta^2(\sin^2\theta + \cos^2\theta) = \boxed{\left(\frac{\partial w}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial w}{\partial \theta}\right)^2}$$

(51) $$F(x) = \int_0^{x^2}\sqrt{t^4 + x^3}\,dt \text{ ; FIND } F'(x).$$

LET $u = x^2$, AND LET $$z = G(u,x) = \int_0^u \sqrt{t^4 + x^3}\,dt.$$



THEN $$F'(x) = \frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial x}\cdot\frac{dx}{dx}$$

$$= \sqrt{u^4 + x^3}\cdot\frac{du}{dx} + \int_0^u \frac{\partial}{\partial x}\left(\sqrt{t^4 + x^3}\right)dt \cdot 1$$

$$= \boxed{\sqrt{x^8 + x^3}\cdot 2x + \int_0^{x^2}\frac{3x^2}{2\sqrt{t^4 + x^3}}\,dt}$$

(USING PART I OF THE FTC IN THE FIRST TERM, AND THE REMARK PRECEDING THE PROBLEM IN THE SECOND TERM)

(8) $f(x,y,z) = 2z^3 - 3(x^2+y^2)z + \tan^{-1} xz$; $(1,1,1)$

$\vec{\nabla} f = \langle f_x, f_y, f_z \rangle = \langle -6xz + \frac{z}{1+(xz)^2}, -6yz, 6z^2 - 3(x^2+y^2) + \frac{x}{1+(xz)^2} \rangle$,

so $\vec{\nabla} f(1,1,1) = \boxed{\langle -\frac{11}{2}, -6, \frac{1}{2} \rangle}$

(18) $h(x,y,z) = \cos xy + e^{yz} + \ln zx$; $P(1,0,\frac{1}{2})$; $\vec{u} = \langle 1,2,2 \rangle$

$\vec{\nabla} h = \langle h_x, h_y, h_z \rangle = \langle -\sin xy \cdot y + \frac{\cancel{z}}{\cancel{z}x}, -\sin xy \cdot x + e^{yz} \cdot z, e^{yz} \cdot y + \frac{\cancel{x}}{z\cancel{x}} \rangle$,

so $\vec{\nabla} h(1,0,\frac{1}{2}) = \langle 1, \frac{1}{2}, 2 \rangle$ if $\vec{v} = \frac{\vec{u}}{|\vec{u}|} = \frac{\langle 1,2,2 \rangle}{\sqrt{1^2+2^2+2^2}} = \langle \frac{1}{3}, \frac{2}{3}, \frac{2}{3} \rangle$,

$D_{\vec{v}} h(1,0,\frac{1}{2}) = (\vec{\nabla} h(1,0,\frac{1}{2})) \cdot \vec{v} = \frac{1}{3} + \frac{1}{3} + \frac{4}{3} = \boxed{2}$

(20) $f(x,y) = x^2 y + e^{xy} \sin y$; $P(1,0)$

$\vec{\nabla} f = \langle f_x, f_y \rangle = \langle 2xy + (e^{xy} \cdot y)\sin y, x^2 + e^{xy}\cos y + (e^{xy} \cdot x)\sin y \rangle$,

so $\vec{\nabla} f(1,0) = \langle 0, 2 \rangle$.

a) increases most rapidly in the direction of $\vec{u} = \frac{\langle 0,2 \rangle}{\sqrt{0^2+2^2}} = \boxed{\langle 0,1 \rangle}$,

and $D_{\vec{u}} f(1,0) = |\vec{\nabla} f(1,0)| = \boxed{2}$

b) decreases most rapidly in the direction of $-\vec{u} = \boxed{\langle 0,-1 \rangle}$,

and $D_{(-\vec{u})} f(1,0) = -|\vec{\nabla} f(1,0)| = \boxed{-2}$

(22) $g(x,y,z) = xe^y + z^2$; $P(1, \ln 2, \frac{1}{2})$

$\vec{\nabla} g = \langle g_x, g_y, g_z \rangle = \langle e^y, xe^y, 2z \rangle$, so $\vec{\nabla} g(1, \ln 2, \frac{1}{2}) = \langle 2, 2, 1 \rangle$

a) increases most rapidly in the direction of $\vec{u} = \frac{\vec{\nabla} g}{|\vec{\nabla} g|} = \frac{\langle 2,2,1 \rangle}{3} = \boxed{\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3} \rangle}$,

and $D_{\vec{u}} g(1, \ln 2, \frac{1}{2}) = |\vec{\nabla} g(1, \ln 2, \frac{1}{2})| = \boxed{3}$

b) decreases most rapidly in the direction of $-\vec{u} = \boxed{\langle -\frac{2}{3}, -\frac{2}{3}, -\frac{1}{3} \rangle}$,

and $D_{(-\vec{u})} g(1, \ln 2, \frac{1}{2}) = -|\vec{\nabla} g(1, \ln 2, \frac{1}{2})| = \boxed{-3}$

(29) $f(x,y) = x^2 - xy + y^2 - y$, $P(1,-1)$

$\vec{\nabla} f = \langle 2x - y, -x + 2y - 1 \rangle$, so $\vec{\nabla} f(1,-1) = \langle 3, -4 \rangle$

a) $D_{\vec{u}} f(-1,1)$ is largest for $\vec{u} = \frac{\langle 3,-4 \rangle}{\sqrt{3^2+4^2}} = \boxed{\langle \frac{3}{5}, -\frac{4}{5} \rangle}$, and then $D_{\vec{u}} f = |\vec{\nabla} f| = \boxed{5}$

b) $D_{\vec{u}} f(-1,1)$ is smallest for $\vec{u} = \boxed{\langle -\frac{3}{5}, \frac{4}{5} \rangle}$, and then $D_{\vec{u}} f = -|\vec{\nabla} f| = \boxed{-5}$

c) $D_{\vec{u}} f(-1,1) = 0$ if $\vec{u}$ is orthogonal to $\langle \frac{3}{5}, -\frac{4}{5} \rangle$, so $\vec{u} = \boxed{\langle \frac{4}{5}, \frac{3}{5} \rangle}$ or $\vec{u} = \boxed{\langle -\frac{4}{5}, -\frac{3}{5} \rangle}$

d) $D_{\vec{u}} f(-1,1) = 4$ iff $\vec{\nabla} f \cdot \vec{u} = 4$ iff $3u_1 - 4u_2 = 4$ (where $\vec{u} = \langle u_1, u_2 \rangle$)

since $u_1^2 + u_2^2 = 1$ and $u_1 = \frac{4u_2 + 4}{3} = \frac{4}{3}(u_2 + 1)$,* $\frac{16}{9}(u_2^2 + 2u_2 + 1) + u_2^2 = 1$ so

$16u_2^2 + 32u_2 + 16 + 9u_2^2 = 9$, $25u_2^2 + 32u_2 + 7 = 0$, $(25u_2 + 7)(u_2 + 1) = 0$, $u_2 = -\frac{7}{25}$ or $u_2 = -1$

so $\vec{u} = \boxed{\langle \frac{24}{25}, -\frac{7}{25} \rangle}$ or $\vec{u} = \boxed{\langle 0, -1 \rangle}$ *(or use $u_2 = \frac{3}{4}u_1 - 1$)

e) $D_{\vec{u}} f(-1,1) = -3$ iff $\vec{\nabla} f \cdot \vec{u} = -3$ iff $3u_1 - 4u_2 = -3$ iff $u_1 = \frac{4u_2 - 3}{3}$. Then $u_1^2 + u_2^2 = 1$

gives $\frac{16u_2^2 - 24u_2 + 9}{9} + u_2^2 = 1$, so $25u_2^2 - 24u_2 + \cancel{9} = \cancel{9}$ and $u_2(25u_2 - 24) = 0$.

Then $u_2 = 0$ or $u_2 = \frac{24}{25}$, so $\vec{u} = \boxed{\langle -1, 0 \rangle}$ or $\vec{u} = \boxed{\langle \frac{7}{25}, \frac{24}{25} \rangle}$