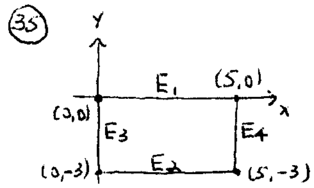
MATH 21C

SOLUTIONS – SEC. 14.7

(35)



$T(x,y) = x^2 + xy + y^2 - 6x + 2$

a) $T_x = 2x + y - 6 = 0$

$T_y = x + 2y = 0$ so $x = -2y$, $2(-2y) + y - 6 = 0$, $-3y = 6$, $\underline{y = -2}$; $\underline{x = 4}$.

THEREFORE $\underline{(4,-2)}$ IS THE ONLY CRITICAL POINT IN THE INTERIOR.

b) ON THE BOUNDARY, T COULD HAVE AN EXTREMUM AT THE 4 VERTICES; AND

1) ON E_1, $y = 0$: LET $f(x) = T(x,0) = x^2 - 6x + 2$. THEN $f'(x) = 2x - 6 = 0$ IF $\underline{x = 3}$: $\underline{(3,0)}$

2) ON E_2, $y = -3$: LET $f(x) = T(x,-3) = x^2 - 9x + 11$. THEN $f'(x) = 2x - 9 = 0$ IF $\underline{x = 9/2}$: $\underline{(9/2, -3)}$

3) ON E_3, $x = 0$: LET $g(y) = y^2 + 2$. THEN $g'(y) = 2y = 0$ IF $y = 0$.

4) ON E_4, $x = 5$: LET $g(y) = y^2 + 5y - 3$. THEN $g'(y) = 2y + 5 = 0$ IF $\underline{y = -5/2}$: $\underline{(5, -5/2)}$

c) $T(0,0) = 2$

$\boxed{T(0,-3) = 11 \text{ IS THE MAX.}}$

$T(5,0) = -3$

$T(5,-3) = -9$

$\boxed{T(4,-2) = -10 \text{ IS THE MIN.}}$

$T(3,0) = -7$

$T(\frac{9}{2}, -3) = -37/4$

$T(5, -5/2) = -37/4$

(56) FIND THE MINIMUM DISTANCE FROM THE CONE $z = \sqrt{x^2 + y^2}$ TO THE POINT $(-6, 4, 0)$.

WE WANT TO MINIMIZE $\underline{d^2 = (x+6)^2 + (y-4)^2 + (z-0)^2}$ WHERE $z = \sqrt{x^2 + y^2}$,

SO LET $\underline{f(x,y) = (x+6)^2 + (y-4)^2 + (x^2 + y^2)}$.

A) $f_x = 2(x+6) + 2x = 0$ GIVES $\underline{x = -3}$

$f_y = 2(y-4) + 2y = 0$ GIVES $\underline{y = 2}$ THEN $z = \sqrt{(-3)^2 + 2^2}$, SO $\underline{z = \sqrt{13}}$

B) $f_{xx} = 4$, $f_{xy} = 0$, $f_{yy} = 4$ SO

$D = 4(4) - 0^2 = 16 > 0$ AND $f_{xx} = 4 > 0$; AND

THEREFORE f HAS A LOCAL MIN. AT $(-3, 2)$:

$d = \sqrt{(-3+6)^2 + (2-4)^2 + (\sqrt{13} - 0)^2} = \sqrt{9 + 4 + 13} = \boxed{\sqrt{26}}$

[SINCE $f(x,y) = x^2 + 12x + 36 + y^2 - 8y + 16 + x^2 + y^2$

$= 2x^2 + 12x + 2y^2 - 8y + 52$

$= 2(x^2 + 6x + \underline{9}) + 2(y^2 - 4y + \underline{4}) + 52 \underline{-18} - \underline{8}$

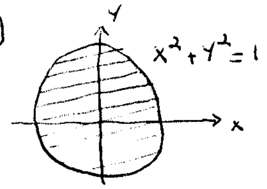
$= \underline{2(x+3)^2 + 2(y-2)^2 + 26}$,

f HAS AN ABSOLUTE MIN. AT $(-3, 2)$.]

MATH 21C

SOLUTIONS – SEC. 14.7

(41)



$T(x,y) = x^2 + 2y^2 - x$

a) $T_x = 2x - 1 = 0$ IF $\underline{x = \frac{1}{2}}$ SO $\underline{\left(\frac{1}{2}, 0\right)}$ IS THE ONLY CRITICAL POINT

$T_y = 4y = 0$ IF $\underline{y = 0}$ (AND IT'S IN THE INTERIOR OF THE REGION)

b) ON THE BOUNDARY $x^2+y^2=1$, LET $\underline{x = \cos\theta,\ y = \sin\theta;\ 0 \le \theta \le 2\pi}$ TO GET

$\underline{T(\theta)} = \cos^2\theta + 2\sin^2\theta - \cos\theta = (\cos^2\theta + \sin^2\theta) + \sin^2\theta - \cos\theta = \underline{1 + \sin^2\theta - \cos\theta}$ FOR $0 \le \theta \le 2\pi$.

THEN $T'(\theta) = 2\sin\theta\cos\theta - (-\sin\theta) = 2\sin\theta\cos\theta + \sin\theta$

$= \sin\theta(2\cos\theta + 1) = 0$ IF $\underline{\sin\theta = 0}$ OR $\underline{\cos\theta = -\frac{1}{2}}$:

1) IF $\underline{\sin\theta = 0}$* AND $0 < \theta < 2\pi$, $\underline{\theta = \pi}$ AND WE GET $\underline{(-1, 0)}$ ⟵ [USING $x = \cos\theta$, $y = \sin\theta$]

2) IF $\underline{\cos\theta = -\frac{1}{2}}$ AND $0 < \theta < 2\pi$, $\underline{\theta = \frac{2\pi}{3}}$ OR $\underline{\theta = \frac{4\pi}{3}}$ AND WE GET $\underline{\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right)}$ AND $\underline{\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)}$

*FOR $\underline{\theta = 0}$ AND $\underline{\theta = 2\pi}$, WE GET THE POINT $\underline{(1, 0)}$.

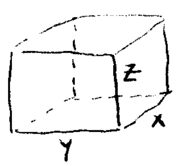
c) $\boxed{T\left(\frac{1}{2}, 0\right) = -\frac{1}{4} \text{ IS THE MIN.}}$

$T(1,0) = 0$

$T(-1,0) = 2$

$\boxed{\begin{array}{l} T\left(-\frac{1}{2}, \frac{\sqrt{3}}{2}\right) = \frac{9}{4} \quad \text{IS THE MAX.} \\ T\left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right) = \frac{9}{4} \end{array}}$

(58)



1) MINIMIZE $\underline{S = 2xy + 2yz + 2xz}$

2) $V = xyz = 27$, SO $\underline{z = \frac{27}{xy}}$ AND

$\underline{S} = 2xy + 2y\left(\frac{27}{xy}\right) + 2x\left(\frac{27}{xy}\right) = \underline{2xy + \frac{54}{x} + \frac{54}{y}}$

3) $S_x = 2y - \frac{54}{x^2} = 0$ SO $\underline{y = \frac{27}{x^2}}$

$S_y = 2x - \frac{54}{y^2} = 0$ SO $\underline{x = \frac{27}{y^2}}$

* THEN $x^2y = 27 = xy^2$, SO (SINCE $x, y \ne 0$) $\underline{x = y}$ AND THEREFORE

$x^3 = 27$, $\underline{x = 3\text{ cm}}$, $\underline{y = 3\text{ cm}}$, AND $\underline{z} = \frac{27}{3 \cdot 3} = \underline{3}\text{ cm}$.

THEREFORE $S = 6(3^2) = \boxed{54\text{ cm}^2}$

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* (OR SUBSTITUTE $y = \frac{27}{x^2}$ INTO $x = \frac{27}{y^2}$ TO GET

$x = 27\left(\frac{1}{y}\right)^2 = 27\left(\frac{x^2}{27}\right)^2 = 27 \cdot \frac{x^4}{27^2} = \frac{x^4}{27}$, SO

$27x = x^4$ AND $27 = x^3$ (SINCE $x \ne 0$) SO $\underline{x = 3\text{ cm}}$)

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① $f(x,y) = x^2 + 2xy + 7y^2 - x^2y - 2xy^2 - 2y^3$

A) $f_x = \underline{2x + 2y - 2xy - 2y^2 = 0}$ so $2x(1-y) + 2y(1-y) = 0$, $2(1-y)(x+y) = 0$, $\underline{y = 1}$ or $\underline{y = -x}$

$f_y = \underline{2x + 14y - x^2 - 4xy - 6y^2 = 0}$

1) If $\underline{y=1}$, $2x + 14 - x^2 - 4x - 6 = 0$ so $0 = x^2 + 2x - 8$, $(x+4)(x-2) = 0$, $\underline{x = -4}$ or $\underline{x = 2}$

2) If $\underline{y = -x}$, $2x - 14x - x^2 + 4x^2 - 6x^2 = 0$ so $0 = 3x^2 + 12x$, $3x(x+4) = 0$, $\underline{x = 0}$ so $\underline{y = 0}$ or $\underline{x = -4}$, $\underline{y = 4}$

CRITICAL POINTS: $\boxed{(-4,1),\ (2,1),\ (0,0),\ (-4,4)}$

B) $f_{xx} = 2 - 2y$ $\quad f_{xy} = 2 - 2x - 4y$ $\quad f_{yy} = 14 - 4x - 12y$

	f_{xx}	f_{xy}	f_{yy}	D
(-4,1)	0	6	18	-36
(2,1)	0	-6	-6	-36
(0,0)	2	2	14	24
(-4,4)	-6	-6	-18	72

SADDLE PT. AT (-4,1)
SADDLE PT. AT (2,1)
LOCAL MIN. AT (0,0)
LOCAL MAX. AT (-4,4)

② $f(x,y) = 4x^2e^y - 2x^4 - e^{4y}$

A) $f_x = \underline{8xe^y - 8x^3 = 0}$ so $8x(e^y - x^2) = 0$, $\underline{x = 0}$ or $\underline{x^2 = e^y}$

$f_y = \underline{4x^2e^y - 4e^{4y} = 0}$ so $4e^y(x^2 - e^{3y}) = 0$, $\underline{x^2 = e^{3y}}$ (since $e^y > 0$ for all y)

1) If $\underline{x = 0}$, $0 = e^{3y}$, which is impossible since $e^{3y} > 0$ for all y

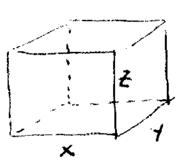
2) If $\underline{x^2 = e^y}$, $e^y = e^{3y}$ gives $y = 3y$, $0 = 2y$, $\underline{y = 0}$ and $x^2 = 1$ so $\underline{x = \pm 1}$

CRITICAL POINTS: $\boxed{(1,0) \text{ AND } (-1,0)}$

B) $f_{xx} = 8e^y - 24x^2$ $\quad f_{xy} = 8xe^y$ $\quad f_{yy} = 4x^2e^y - 16e^{4y}$

	f_{xx}	f_{xy}	f_{yy}	D
(1,0)	-16	8	-12	128
(-1,0)	-16	-8	-12	128

LOCAL MAX. AT (1,0)
LOCAL MAX. AT (-1,0)

③ 

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1) Minimize $\underline{S = xy + 2xz + 2yz}$

2) $\underline{V = xyz = 32}$, so $\underline{z = \frac{32}{xy}}$ and therefore

$S = xy + 2x\left(\frac{32}{xy}\right) + 2y\left(\frac{32}{xy}\right) = \underline{xy + \frac{64}{y} + \frac{64}{x}}$

3) $S_x = \underline{y - \frac{64}{x^2} = 0}$ so $y = \frac{64}{x^2}$

$S_y = \underline{x - \frac{64}{y^2} = 0}$ so $x = \frac{64}{y^2}$

$x = 64 \cdot \frac{1}{y^2} = 64\left(\frac{1}{y}\right)^2 = 64\left(\frac{x^2}{64}\right)^2 = \cancel{64} \cdot \frac{x^4}{64^{\cancel{2}}} = \frac{x^4}{64}$ so $64x = x^4$,

$x^3 = 64$ (since $x \neq 0$) and $\boxed{x = 4 \text{ cm}}$

Then $y = \frac{64}{x^2}$ gives $\boxed{y = 4 \text{ cm}}$, and $z = \frac{32}{xy}$ gives $\boxed{z = 2 \text{ cm}}$

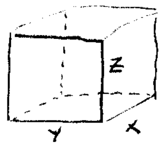
4) $S_{xx} = \frac{128}{x^3}$, $S_{xy} = 1$, $S_{yy} = \frac{128}{y^3}$

at (4,4), $\underline{S_{xx} = 2}$, $S_{xy} = 1$, $S_{yy} = 2$, and $\underline{D = 3}$;

so S has a local $\underline{\text{min.}}$ at (4,4).

MATH 21C SOLUTIONS – WHITE SHEET (SEC. 14.7 AND 14.8)

(4)



1) MINIMIZE COST $\underline{C} = 2(XY) + 1(XY) + .2(2YZ) + .2(2XZ)$,

SO $\underline{C = 3XY + .4YZ + .4XZ}$

2) $\underline{V = XYZ = 60}$, SO $\underline{Z = \frac{60}{XY}}$ AND

$\underline{C} = 3XY + .4\cancel{Y}\left(\frac{60}{X\cancel{Y}}\right) + .4\cancel{X}\left(\frac{60}{\cancel{X}Y}\right) = \underline{3XY + \frac{24}{X} + \frac{24}{Y}}$

3) $C_X = 3Y - \frac{24}{X^2} = 0$ SO $\underline{Y = \frac{8}{X^2}}$

$C_Y = 3X - \frac{24}{Y^2} = 0$ SO $\underline{X = \frac{8}{Y^2}}$

$X^2Y = 8 = XY^2$, SO $\underline{X = Y}$ (SINCE $\underline{X \neq 0}$ AND $\underline{Y \neq 0}$)

AND $X = \frac{8}{X^2}$ GIVES $X^3 = 8$ SO $\boxed{X = 2 \text{ FT}}$

THEN $Y = \frac{8}{X^2}$ GIVES $\boxed{Y = 2 \text{ FT}}$, AND $Z = \frac{60}{XY}$ GIVES $\boxed{Z = 15 \text{ FT}}$

4) $C_{XX} = \frac{48}{X^3}$ $\quad C_{XY} = 3$ $\quad C_{YY} = \frac{48}{Y^3}$

	C_{XX}	C_{XY}	C_{YY}	D
(2,2)	6	3	6	27

LOCAL $\underline{\text{MIN.}}$ AT (2,2)

(5) $f(x,y) = x^2 - y^2 + 2xy$ ON THE CLOSED DISC BOUNDED BY $x^2 + y^2 = 9$.

A) $f_x = \underline{2x + 2y = 0}$ SO $y = -x$

$f_y = \underline{-2y + 2x = 0}$ SO $y = x$

$x = -x$, $2x = 0$, $\underline{x = 0}$, $\underline{y = 0}$

SO $\underline{(0,0)}$ IS THE ONLY CRITICAL POINT (AND IT IS IN THE INTERIOR OF THE REGION).

B) ON THE BOUNDARY $x^2 + y^2 = 9$, LET $\underline{x = 3\cos\theta,\ y = 3\sin\theta}$, θ IN $[0, 2\pi)$ TO GET

$\underline{g(\theta)} = (3\cos\theta)^2 - (3\sin\theta)^2 + 2(3\cos\theta)(3\sin\theta) = 9\left(\underline{\cos^2\theta - \sin^2\theta}\right) + 9\left(\underline{2\sin\theta\cos\theta}\right)$,

SO $\underline{g(\theta)} = 9(\cos 2\theta) + 9(\sin 2\theta) = \underline{9(\cos 2\theta + \sin 2\theta)}$.

THEN $g'(\theta) = 9(-2\sin 2\theta + 2\cos 2\theta) = 0$ IF $\cancel{2}\sin 2\theta = \cancel{2}\cos 2\theta$, $\frac{\sin 2\theta}{\cos 2\theta} = 1$,

$\tan 2\theta = 1$, $\underline{2\theta} = \frac{\pi}{4}, \frac{5\pi}{4}, \frac{9\pi}{4}, \frac{13\pi}{4}$ AND $\underline{\theta} = \frac{\pi}{8}, \frac{5\pi}{8}, \frac{9\pi}{8}, \frac{13\pi}{8}$

C)

1) $f(0,0) = \underline{0}$

2) $\theta = \frac{\pi}{8}$: $\boxed{f\left(3\cos\frac{\pi}{8}, 3\sin\frac{\pi}{8}\right)} = g\left(\frac{\pi}{8}\right) = 9\left(\cos\frac{\pi}{4} + \sin\frac{\pi}{4}\right) = \boxed{9\sqrt{2}}$

3) $\theta = \frac{9\pi}{8}$: $\boxed{f\left(3\cos\frac{9\pi}{8}, 3\sin\frac{9\pi}{8}\right)} = g\left(\frac{9\pi}{8}\right) = 9\left(\cos\frac{9\pi}{4} + \sin\frac{9\pi}{4}\right) = \boxed{9\sqrt{2}}$

$\longleftarrow$ $\underline{\text{MAX.}}$ VALUE

4) $\theta = \frac{5\pi}{8}$: $\boxed{f\left(3\cos\frac{5\pi}{8}, 3\sin\frac{5\pi}{8}\right)} = g\left(\frac{5\pi}{8}\right) = 9\left(\cos\frac{5\pi}{4} + \sin\frac{5\pi}{4}\right) = \boxed{-9\sqrt{2}}$

5) $\theta = \frac{13\pi}{8}$: $\boxed{f\left(3\cos\frac{13\pi}{8}, 3\sin\frac{13\pi}{8}\right)} = g\left(\frac{13\pi}{8}\right) = 9\left(\cos\frac{13\pi}{4} + \sin\frac{13\pi}{4}\right) = \boxed{-9\sqrt{2}}$

$\longleftarrow$ $\underline{\text{MIN.}}$ VALUE

$\underline{\text{REMARK}}$ WE COULD USE THE HALF-ANGLE FORMULAS

$\cos\frac{\theta}{2} = \pm\sqrt{\frac{1+\cos\theta}{2}}$ AND $\sin\frac{\theta}{2} = \pm\sqrt{\frac{1-\cos\theta}{2}}$

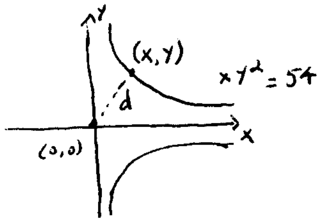
TO WRITE THE COORDINATES OF THESE POINTS WITHOUT TRIG FUNCTIONS:

FOR EXAMPLE, $\cos\frac{\pi}{8} = \frac{\sqrt{2+\sqrt{2}}}{2}$ AND $\sin\frac{\pi}{8} = \frac{\sqrt{2-\sqrt{2}}}{2}$.

MATH 21C

SOLUTIONS – Sec. 14.8

(5) FIND THE POINTS ON THE CURVE $xy^2 = 54$ NEAREST THE ORIGIN.



MINIMIZE $f(x,y) = d^2 = (x-0)^2 + (y-0)^2 = x^2+y^2$, SUBJECT TO $\underbrace{xy^2}_{g(x,y)} = 54$

$2x = \lambda(y^2)$ SO $\lambda = \frac{2x}{y^2}$ AND $\frac{2x}{y^2} = \frac{1}{x}$, $y^2 = 2x^2$

$2y = \lambda(2xy)$ $\quad\lambda = \frac{1}{x}$

SUBSTITUTING INTO THE CONSTRAINT GIVES $x(2x^2) = 54$, $2x^3 = 54$, $x^3 = 27$, $x = 3$ AND

$y^2 = 2(3^2)$ SO $y = \pm 3\sqrt{2}$

THUS $\boxed{(3, 3\sqrt{2}) \text{ AND } (3, -3\sqrt{2})}$ ARE THE POINTS CLOSEST TO THE ORIGIN.

REMARK SUBSTITUTING $y^2 = \frac{54}{x}$ INTO $f(x,y) = x^2+y^2$ GIVES A WAY TO REDUCE THIS TO A PROBLEM INVOLVING A FUNCTION OF 1 VARIABLE.

(8) MAX. AND MIN. OF $f(x,y) = d^2 = x^2+y^2$ ON $\underbrace{x^2+xy+y^2}_{g(x,y)} = 1$

$2x = \lambda(2x+y)$ SO $\frac{1}{\lambda} = 1 + \frac{y}{2x}$ AND $1 + \frac{y}{2x} = 1 + \frac{x}{2y}$, $\frac{y}{2x} = \frac{x}{2y}$, $2y^2 = 2x^2$,

$2y = \lambda(x+2y)$ $\quad\frac{1}{\lambda} = 1 + \frac{x}{2y}$ $\quad$ SO $y^2 = x^2$ AND $y = \pm x$.

1) IF $y = x$, $x^2+x^2+x^2 = 1$, $3x^2 = 1$, $x^2 = \frac{1}{3}$, $x = \pm\frac{1}{\sqrt{3}}$: $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$ AND $\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$

2) IF $y = -x$, $x^2 - x^2 + x^2 = 1$, $x^2 = 1$, $x = \pm 1$: $(1,-1)$ AND $(-1,1)$

$f\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3} = f\left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$ AND $f(1,-1) = 2 = f(-1,1)$,

SO $\boxed{\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right) \text{ AND } \left(-\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)}$ ARE THE CLOSEST POINTS TO $(0,0)$

AND $\boxed{(1,-1) \text{ AND } (-1,1)}$ ARE THE FARTHEST POINTS FROM $(0,0)$

(13) FIND THE EXTREMA OF $f(x,y) = x^2+y^2$ ON THE CIRCLE $\underbrace{x^2-2x+y^2-4y}_{g(x,y)} = 0$.

$2x = \lambda(2x-2)$ SO $\frac{1}{\lambda} = 1 - \frac{1}{x}$ AND $1 - \frac{1}{x} = 1 - \frac{2}{y}$, $\frac{2}{y} = \frac{1}{x}$, $y = 2x$

$2y = \lambda(2y-4)$ $\quad\frac{1}{\lambda} = 1 - \frac{2}{y}$

THEN $x^2 - 2x + 4x^2 - 8x = 0$, $5x^2 - 10x = 0$, $5x(x-2) = 0$, $\begin{matrix} x=0 \\ y=0 \end{matrix}$ OR $\begin{matrix} x=2 \\ y=4 \end{matrix}$

$$\boxed{\begin{matrix} f(0,0) = 0 \text{ IS THE MIN,} \\ f(2,4) = 20 \text{ IS THE MAX.} \end{matrix}}$$

REMARK SINCE $f(x,y) = d^2$ WHERE d IS THE DISTANCE FROM (x,y) TO $(0,0)$, $(0,0)$ AND $(2,4)$ ARE THE POINTS ON THE CIRCLE CLOSEST TO, AND FARTHEST FROM, THE ORIGIN.