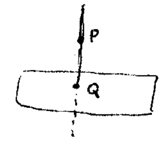
MATH 21C

SOLUTIONS – WHITE SHEET (Sec. 14.7 – 14.8)

6 $3x + 2y - z = -17$; FIND POINT CLOSEST TO $P(5,2,8)$

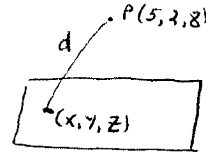


A) THE LINE THROUGH P PERPENDICULAR TO THE PLANE IS GIVEN BY

$\underline{x = 5 + 3t,\ y = 2 + 2t,\ z = 8 - t}$; SO THE LINE AND PLANE INTERSECT

WHERE $3(5+3t) + 2(2+2t) - (8-t) = -17$, $14t = -28$, $\underline{t = -2}$:

$\underline{x = -1,\ y = -2,\ z = 10}$; SO $\boxed{(-1,-2,10)}$ IS THE CLOSEST POINT.



B) MINIMIZE $\underline{d^2 = (x-5)^2 + (y-2)^2 + (z-8)^2}$ WHERE $\underline{z = 3x + 2y + 17}$,

SO LET $\underline{f(x,y) = (x-5)^2 + (y-2)^2 + (3x+2y+9)^2}$.

1) $f_x = 2(x-5) + 2(3x+2y+9)\cdot 3 = 0$ SO $10x + 6y = -22$

$f_y = 2(y-2) + 2(3x+2y+9)\cdot 2 = 0$ SO $6x + 5y = -16$

$$\begin{array}{rrl} & 50x + 30y &= -110 \\ \ominus & 36x + 30y &= -96 \\ \hline & 14x &= -14 \end{array}$$

SO $\underline{x = -1}$ AND $5y = -16 + 6 = -10$, $\underline{y = -2}$

AND $\underline{z} = 3(-1) + 2(-2) + 17 = \underline{10}$.

2) $f_{xx} = 20$, $f_{xy} = 12$, $f_{yy} = 10$ SO $D = 20(10) - 12^2 = 56 > 0$ AND $f_{xx} = 20 > 0$;

SO f HAS A LOCAL MIN. AT $(-1,-2)$, THEREFORE $\boxed{(-1,-2,10)}$ IS THE CLOSEST POINT.

C) MINIMIZE $\underline{f(x,y,z) = (x-5)^2 + (y-2)^2 + (z-8)^2}$ ON THE PLANE $\underbrace{3x + 2y - z}_{g(x,y,z)} = -17$:

$2(x-5) = \lambda\cdot 3$ $\quad\lambda = \frac{2}{3}(x-5)$

$2(y-2) = \lambda\cdot 2$ $\quad\lambda = y - 2$

$2(z-8) = \lambda(-1)$ $\quad\lambda = -2(z-8)$

$\frac{2}{3}(x-5) = -2(z-8)$, $x - 5 = -3z + 24$, $\underline{x = -3z + 29}$

$y - 2 = -2(z-8) = -2z + 16$, $\underline{y = -2z + 18}$.

SUBSTITUTING INTO THE CONSTRAINT GIVES $3(-3z+29) + 2(-2z+18) - z = -17$,

SO $-14z = -140$, $\underline{z = 10}$, $\underline{x = -1}$, $\underline{y = -2}$: $\boxed{(-1,-2,10)}$

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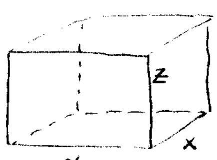

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MAXIMIZE $\underline{V = xyz}$, SUBJECT TO $S = \underline{xy + 2yz + 2xz = 48}$:

1) $yz = \lambda(y + 2z)$

2) $xz = \lambda(x + 2z)$

3) $xy = \lambda(2y + 2x)$

MULTIPLYING BY x IN 1), y IN 2), AND z IN 3) GIVES

1) $xyz = \underline{\lambda(xy + 2xz)}$, 2) $xyz = \underline{\lambda(xy + 2yz)}$, 3) $xyz = \underline{\lambda(2yz + 2xz)}$.

a) THEN $\lambda(xy + 2xz) = \lambda(xy + 2yz)$ WHERE $\lambda \neq 0$ (SINCE $x, y, z > 0$); SO

$\cancel{xy} + 2xz = \cancel{xy} + 2yz$, $2xz = 2yz$, $\underline{y = x}$ (SINCE $z > 0$).

b) ALSO, $\lambda(xy + 2xz) = \lambda(2yz + 2xz)$ WITH $\lambda \neq 0$, SO

$xy + \cancel{2xz} = 2yz + \cancel{2xz}$, $xy = 2yz$, $\underline{z = \frac{x}{2}}$ (SINCE $y > 0$).

SUBSTITUTING INTO THE CONSTRAINT GIVES

$x^2 + 2x\left(\frac{x}{2}\right) + 2x\left(\frac{x}{2}\right) = 48$, SO $3x^2 = 48$, $x^2 = 16$, $\boxed{x = 4\text{ FT},\ y = 4\text{ FT},\ z = 2\text{ FT}}$

(9) $f(x,y) = xy$ on the ellipse $\underbrace{3x^2+4x+4y^2}_{g(x,y)} = 0$.

$y = \lambda(6x+4)$ $\quad \lambda = \frac{y}{6x+4}$ $\quad$ so $\frac{y}{6x+4} = \frac{x}{8y}$, $\quad 8y^2 = 6x^2+4x$, $\quad \underline{4y^2 = 3x^2+2x}$ so (substituting into the constraint)

$x = \lambda(8y)$ $\quad \lambda = \frac{x}{8y}$

$3x^2+4x+(3x^2+2x) = 0$, $\quad 6x^2+6x=0$, $\quad 6x(x+1)=0$, $\quad \underline{x=0}$ or $\underline{x=-1}$

if $\underline{x=0}$, $\quad 4y^2=0$ so $\underline{y=0}$

if $\underline{x=-1}$, $\quad 4y^2 = 1$ so $y^2 = \frac{1}{4}$ and $\underline{y = \pm\frac{1}{2}}$

1) $f(0,0) = 0$

2) $f(-1,\frac{1}{2}) = -\frac{1}{2}$

3) $\boxed{f(-1,-\frac{1}{2}) = \frac{1}{2}}$ is the max. value

(10) $f(x,y,z) = 2x-3y+z$ on the ellipsoid $\underbrace{(x-5)^2+3y^2+2(z+4)^2}_{g(x,y,z)} = 30$.

$2 = \lambda\cdot 2(x-5)$ $\quad \frac{1}{\lambda} = x-5$ $\quad x-5 = -2y$, $\quad \underline{x = -2y+5}$ so (substituting into the constraint)

$-3 = \lambda\cdot 6y$ $\quad \frac{1}{\lambda} = -2y$ $\quad 4z+16 = -2y$, $\quad \underline{z = -\frac{1}{2}y - 4}$ *

$1 = \lambda\cdot 4(z+4)$ $\quad \frac{1}{\lambda} = 4(z+4)$

$(-2y)^2 + 3y^2 + 2(-\frac{1}{2}y)^2 = 30$, $\quad 4y^2+3y^2+\frac{1}{2}y^2 = 30$, $\quad \frac{15}{2}y^2 = 30$, $\quad y^2=4$, $\quad \underline{y=\pm 2}$

1) if $\underline{y=2}$, $\quad \boxed{f(1,2,-5) = -9}$ is the min.

2) if $\underline{y=-2}$, $\quad \boxed{f(9,-2,-3) = 21}$ is the max.

14.8 - (23) max. and min. values of $f(x,y,z) = x-2y+5z$ on the sphere $\underbrace{x^2+y^2+z^2}_{g(x,y,z)} = 30$.

$1 = \lambda\cdot 2x$ $\quad \frac{1}{\lambda} = 2x$ $\quad$ so $-y = 2x$, $\quad \underline{y=-2x}$

$-2 = \lambda\cdot 2y$ $\quad \frac{1}{\lambda} = -y$ $\quad \frac{2}{5}z = 2x$, $\quad \underline{z = 5x}$

$5 = \lambda\cdot 2z$ $\quad \frac{1}{\lambda} = \frac{2}{5}z$

Then $x^2 + (-2x)^2 + (5x)^2 = 30$, $\quad x^2+4x^2+25x^2 = 30$, $\quad 30x^2 = 30$, $\quad x^2=1$, $\quad \underline{x=\pm 1}$

1) $\underline{x=1}$: $\quad \boxed{f(1,-2,5) = 30}$ is the max.

2) $\underline{x=-1}$: $\quad \boxed{f(-1,2,-5) = -30}$ is the min.

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* OR use $x-5 = 4(z+4)$ and $y = -2(z+4)$ to get

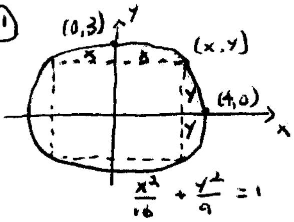
$16(z+4)^2 + 3\cdot 4(z+4)^2 + 2(z+4)^2 = 30$, so $30(z+4)^2 = 30$, $\quad (z+4)^2 = 1$,

$z+4 = \pm 1$, $\quad \underline{z=-3}$ or $\underline{z=-5}$

MATH 21C

SOLUTIONS – Sec. 14.8

(11)



MAXIMIZE $A = (2x)(2y) = 4xy$ SUBJECT TO $\underbrace{9x^2+16y^2}_{g(x,y)} = 144$

$4y = \lambda(18x)$ so $\lambda = \frac{2y}{9x}$

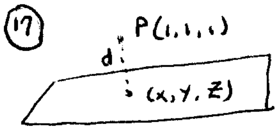
$4x = \lambda(32y)$ $\quad \lambda = \frac{x}{8y}$ AND $\frac{2y}{9x} = \frac{x}{8y}$, $9x^2 = 16y^2$

SUBSTITUTING INTO THE CONSTRAINT GIVES $16y^2+16y^2 = 144$, $32y^2 = 144$, $y^2 = \frac{9}{2}$, $y = \frac{3}{\sqrt{2}}$ (SINCE $y>0$)

THEN $x^2 = \frac{16}{9}y^2 = \frac{16}{9}\cdot\frac{9}{2} = 8$, $x = 2\sqrt{2}$ (SINCE $x>0$)

THEREFORE THE RECTANGLE HAS BASE $b = 2x = \boxed{4\sqrt{2}}$ AND HEIGHT $h = 2y = \boxed{3\sqrt{2}}$.

(17)



MINIMIZE $f(x,y,z) = d^2 = (x-1)^2+(y-1)^2+(z-1)^2$,

SUBJECT TO $\underbrace{x+2y+3z}_{g(x,y,z)} = 13$.

$2(x-1) = \lambda\cdot 1$ $\qquad \lambda = 2(x-1)$

$2(y-1) = \lambda\cdot 2$ $\quad$ so $\quad \lambda = y-1$

$2(z-1) = \lambda\cdot 3$ $\qquad \lambda = \frac{2}{3}(z-1)$

THEN $y-1 = 2(x-1)$ SO $y = 2x-1$ AND

$\frac{2}{3}(z-1) = 2(x-1)$, $z-1 = 3(x-1)$, $z = 3x-2$.

SUBSTITUTING INTO THE EQUATION OF THE PLANE GIVES $x+2(2x-1)+3(3x-2) = 13$,

$14x = 21$, $x = \frac{3}{2}$, $y = 2$, $z = \frac{5}{2}$; SO $\boxed{\left(\frac{3}{2}, 2, \frac{5}{2}\right)}$ IS THE CLOSEST POINT.

(20) MINIMIZE $f(x,y,z) = d^2 = x^2+y^2+z^2$ ON $z = xy+1$ OR $\underbrace{z-xy}_{g(x,y,z)} = 1$

$2x = \lambda(-y)$ $\quad \lambda = \frac{2x}{-y}$

$2y = \lambda(-x)$ $\quad \lambda = \frac{2y}{-x}$ $\quad$ so $\frac{2x}{-y} = \frac{2y}{-x}$, $-2x^2 = -2y^2$, $x^2 = y^2$, $y = \pm x$

$2z = \lambda(1)$ $\quad \lambda = 2z$

1) IF $x=0$ AND $y=0$, THEN $z-0=1$ SO $z=1$

2) IF $x\neq 0$ AND $y=x$, THEN $2z = \lambda = \frac{2x}{-x} = -2$ SO $z=-1$.
THEN $-1-x^2 = 1$, SO $x^2 = -2$, WHICH IS IMPOSSIBLE.

3) IF $x\neq 0$ AND $y=-x$, THEN $2z = \lambda = -\frac{2x}{-x} = 2$ SO $z=1$,
THEN $1-xy = 1$, SO $xy=0$ AND $x=0$ OR $y=0$, WHICH GIVES A CONTRADICTION (SINCE $x\neq 0$ AND $y=-x$).

THEREFORE $\boxed{(0,0,1)}$ IS THE CLOSEST POINT TO $(0,0,0)$.

(21) MINIMIZE $f(x,y,z) = d^2 = x^2+y^2+z^2$ ON $z^2 = xy+4$ OR $\underbrace{z^2-xy}_{g(x,y,z)} = 4$.

$2x = \lambda(-y)$ $\quad \lambda = \frac{2x}{-y}$ $\quad$ so $\frac{2x}{-y} = \frac{2y}{-x}$, $-2x^2 = -2y^2$, $y^2 = x^2$, $y = \pm x$

$2y = \lambda(-x)$ $\quad \lambda = \frac{2y}{-x}$

$2z = \lambda(2z)$ $\quad$ so 1) $\lambda = 1$ OR 2) $z=0$

1) IF $\lambda=1$, $2x=-y$ AND $2y=-x$ GIVES $y=-2x$ SO $-4x=-x$, $3x=0$, $x=0$, $y=0$,
AND $z^2=4$ SO $z=\pm 2$

2) IF $z=0$, $-xy=4$ SO $xy=-4$
a) IF $y=x$, THEN $x^2=-4$ (NO SOLUTION)
b) IF $y=-x$, THEN $-x^2=-4$, $x^2=4$, $x=\pm 2$ AND $y=\mp 2$.

i) $f(0,0,2) = 4$ AND ii) $f(0,0,-2) = 4$

iii) $f(2,-2,0) = 8$ AND iv) $f(-2,2,0) = 8$

THEREFORE $\boxed{(0,0,2) \text{ AND } (0,0,-2)}$ ARE THE POINTS CLOSEST TO THE ORIGIN.

MATH 21C SOLUTIONS – SEC. 14.8

(38) MINIMIZE $f(x,y,z) = x^2+y^2+z^2$, SUBJECT TO $\underbrace{x+2y+3z=6}_{g(x,y,z)}$ AND $\underbrace{x+3y+9z=9}_{h(x,y,z)}$.

1) $2x = \lambda\cdot 1 + \mu\cdot 1 = \lambda+\mu$
2) $2y = \lambda\cdot 2 + \mu\cdot 3 = 2\lambda+3\mu$
3) $2z = \lambda\cdot 3 + \mu\cdot 9 = 3\lambda+9\mu$

THEN $\mu = 2y-4x$ (SUBT. 2 TIMES EQ. 1 FROM EQ. 2) AND
$6\mu = 2z-6x$ (SUBT. 3 TIMES EQ. 1 FROM EQ. 3).

THEN $2z-6x = 6(2y-4x)$, SO $24x-6x-12y+2z=0$ AND $9x-6y+z=0$

SUBSTITUTING $z = 6y-9x$ IN THE CONSTRAINT EQUATIONS GIVES

$x+2y+3(6y-9x)=6$ AND $x+3y+9(6y-9x)=9$, SO

$-26x+20y=6$ AND $-80x+57y=9$

SO $-13x+10y=3$ AND $-80x+57y=9$

$$\begin{aligned} 57(-13x+10y) &= 57(3) \\ -\ 10(-80x+57y) &= 10(9) \\ \hline 59x &= 81 \end{aligned}$$

SO $x = \frac{81}{59}$ $\quad 10y = 3+13\left(\frac{81}{59}\right)$ GIVES $y = \frac{123}{59}$ AND $z = 6\left(\frac{123}{59}\right)-9\left(\frac{81}{59}\right) = \frac{9}{59}$

$f\left(\frac{81}{59}, \frac{123}{59}, \frac{9}{59}\right) = \frac{81^2+123^2+9^2}{59^2} = \frac{369}{59}$ IS THE MIN.*

(42a) MAXIMIZE $f(x,y,z) = xyz$, SUBJECT TO $\underbrace{x+y+z=40}_{g(x,y,z)}$ AND $\underbrace{x+y-z=0}_{h(x,y,z)}$

1) $yz = \lambda\cdot 1+\mu\cdot 1 = \lambda+\mu$
2) $xz = \lambda\cdot 1+\mu\cdot 1 = \lambda+\mu$
3) $xy = \lambda\cdot 1+\mu(-1) = \lambda-\mu$

FROM 1) AND 2), $xz = yz$ SO EITHER a) $x=y$ OR b) $z=0$

a) IF $x=y$, $2x+z=40$ AND $2x-z=0$; SO ADDING THESE EQUATIONS GIVES $4x=40$, $x=10$, SO $y=10$ AND $z=20$.

b) IF $z=0$, $x+y=40$ AND $x+y=0$; SO THIS IS IMPOSSIBLE.

THEREFORE $\boxed{f(10,10,20) = 2000}$ IS THE MAX. VALUE.

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* $\boxed{\text{OR}}$ SOLVING FOR x IN THE FIRST EQUATION GIVES $x = 6-2y-3z$, SO SUBSTITUTING INTO THE FUNCTION AND THE SECOND EQUATION GIVES THE EQUIVALENT PROBLEM OF

MINIMIZING $g(y,z) = (6-2y-3z)^2+y^2+z^2$, SUBJECT TO $y+6z=3$.

THEN $2(6-2y-3z)(-2)+2y = \lambda\cdot 1$
AND $2(6-2y-3z)(-3)+2z = \lambda\cdot 6$,

SO $-12+4y+6z+y = \frac{\lambda}{2}$, $\quad 5y+6z-12 = \frac{\lambda}{2}$
$-18+6y+9z+z = 3\lambda$, $\quad 6y+10z-18 = 3\lambda$.

THEN $6y+10z-18 = 6(5y+6z-12)$, SO
$54 = 24y+26z$ AND $12y+13z=27$

SINCE $y = 3-6z$ FROM THE CONSTRAINT, $12(3-6z)+13z=27$
SO $-59z=-9$ AND $z = 9/59$ THEN $y = 3-6\left(\frac{9}{59}\right) = \frac{123}{59}$

AND $x = 6-2\left(\frac{123}{59}\right)-3\left(\frac{9}{59}\right) = \frac{81}{59}$

MATH 21C SOLUTIONS – WHITE SHEET AND Sec. 14.10

(12)' $f(x,y,z) = 4x + 3y + 4z$, subject to $\underbrace{x + y + 2z = 8}_{g(x,y,z)}$ and $\underbrace{x^2 + y^2}_{h(x,y,z)} = 20$

1) $4 = \lambda(1) + \mu(2x)$

2) $3 = \lambda(1) + \mu(2y)$

3) $4 = \lambda(2) + \mu(0)$ so $4 = 2\lambda$, $\underline{\lambda = 2}$ and 1) $4 = 2 + 2\mu x$, $2\mu x = 2$, $\underline{\frac{1}{\mu} = x}$

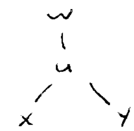
2) $3 = 2 + 2\mu y$, $2\mu y = 1$, $\underline{\frac{1}{\mu} = 2y}$

Then $\underline{x = 2y}$, so $x^2 + y^2 = 20$ gives $4y^2 + y^2 = 20$, $5y^2 = 20$, $y^2 = 4$, $\underline{y = \pm 2}$

a) if $\underline{y = 2}$, then $\underline{x = 4}$ and $6 + 2z = 8$, so $\underline{z = 1}$: $\boxed{f(4,2,1) = \underline{26} \text{ is the MAX.}}$

b) if $\underline{y = -2}$, then $\underline{x = -4}$ and $-6 + 2z = 8$, so $\underline{z = 7}$: $\boxed{f(-4,-2,7) = \underline{6} \text{ is the MIN.}}$ *

14.10 – (10) $z = x + f(u)$ where $u = xy$ Let $\underline{w = f(u)}$, so $\underline{z = x + w}$:



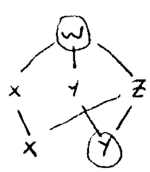
$$\frac{\partial z}{\partial x} = 1 + \frac{\partial w}{\partial x} = 1 + \frac{dw}{du}\cdot\frac{\partial u}{\partial x} = 1 + f'(u)\cdot y = \underline{1 + y f'(xy)}$$

$$\frac{\partial z}{\partial y} = 0 + \frac{\partial w}{\partial y} = \frac{dw}{du}\cdot\frac{\partial u}{\partial y} = f'(u)\cdot x = \underline{x f'(xy)}$$

Then $\underline{x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}} = x\left[1 + y f'(xy)\right] - y\left[x f'(xy)\right] = x + xy f'(xy) - xy f'(xy) = \underline{x}$

(11) Let $w = g(x,y,z)$ with $z = h(x,y)$, so we have that

$w = g(x, y, h(x,y)) = 0$.



Since w is a constant function of x and y, $\underline{\left(\frac{\partial w}{\partial y}\right)_x = 0}$

where $\underline{\left(\frac{\partial w}{\partial y}\right)_x = \left(\frac{\partial w}{\partial y}\right)_{x,z}\cdot\frac{\partial y}{\partial y} + \left(\frac{\partial w}{\partial z}\right)_{x,y}\cdot\left(\frac{\partial z}{\partial y}\right)_x}$,

so $0 = g_y \cdot 1 + g_z \cdot \left(\frac{\partial z}{\partial y}\right)_x$

and $\underline{\left(\frac{\partial z}{\partial y}\right)_x = -\frac{g_y}{g_z}}$.

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* OR Solving for $2z$ in the first equation gives $2z = 8 - x - y$, so substituting into the function gives the equivalent problem of finding the extrema of

$g(x,y) = 4x + 3y + 2(8 - x - y) = 2x + y + 16$, subject to $\underbrace{x^2 + y^2}_{h(x,y)} = 20$:

$2 = \lambda \cdot 2x$ so $\frac{1}{\lambda} = x$ and $\underline{x = 2y}$

$1 = \lambda \cdot 2y$ so $\frac{1}{\lambda} = 2y$

substituting into the constraint gives $4y^2 + y^2 = 20$, $5y^2 = 20$, $y^2 = 4$, $\underline{y = \pm 2}$

1) if $\underline{y = 2}$, $\underline{x = 4}$ and $2z = 2$ so $\underline{z = 1}$: $f(4,2,1) = \underline{26}$ is the MAX.

2) if $\underline{y = -2}$, $\underline{x = -4}$ and $2z = 14$ so $\underline{z = 7}$: $f(-4,-2,7) = \underline{6}$ is the MIN.