MATH 21C SOLUTIONS - SEC. 13.1 AND 13.2

13.1 - 4 $\vec{r}(t) = (\cos 2t)\vec{i} + (3\sin 2t)\vec{j}$, $t = 0$

$\cos^2 2t + \sin^2 2t = 1$, so $x^2 + \left(\frac{y}{3}\right)^2 = 1$ or $\boxed{x^2 + \frac{y^2}{9} = 1}$ (ELLIPSE)

$\vec{v}(t) = \vec{r}\,'(t) = (-2\sin 2t)\vec{i} + (6\cos 2t)\vec{j}$, so $\boxed{\vec{v}(0) = 6\vec{j}}$

$\vec{a}(t) = \vec{v}\,'(t) = (-4\cos 2t)\vec{i} - (12\sin 2t)\vec{j}$, so $\boxed{\vec{a}(0) = -4\vec{i}}$

16 $\vec{r}(t) = \left(\frac{\sqrt{2}}{2}t\right)\vec{i} + \left(\frac{\sqrt{2}}{2}t - 16t^2\right)\vec{j}$, $t = 0$

$\vec{v}(t) = \vec{r}\,'(t) = \frac{\sqrt{2}}{2}\vec{i} + \left(\frac{\sqrt{2}}{2} - 32t\right)\vec{j}$, so $\vec{v}(0) = \frac{\sqrt{2}}{2}\vec{i} + \frac{\sqrt{2}}{2}\vec{j}$

$\vec{a}(t) = \vec{v}\,'(t) = -32\vec{j}$, so $\vec{a}(0) = -32\vec{j}$

SINCE $\vec{v}(0) = \frac{\sqrt{2}}{2}\langle 1,1\rangle$ AND $\vec{a}(0) = 32\langle 0,-1\rangle$,

THE ANGLE θ BETWEEN $\vec{v}(0)$ AND $\vec{a}(0)$ SATISFIES

$$\cos\theta = \frac{\langle 1,1\rangle \cdot \langle 0,-1\rangle}{|\langle 1,1\rangle|\,|\langle 0,-1\rangle|} = \frac{-1}{\sqrt{2}(1)} = -\frac{1}{\sqrt{2}}, \quad \text{so } \theta = \pi - \frac{\pi}{4} = \boxed{\frac{3\pi}{4}}$$

(SINCE $0 \le \theta \le \pi$)

13.2 - 17 LET $P = (1,2,3)$ AND $Q = (4,1,4)$, so $\overrightarrow{PQ} = \langle 3,-1,1\rangle$.

THEN $\vec{r}(0) = \langle 1,2,3\rangle$,

$\vec{v}(0) = 2\left(\frac{\overrightarrow{PQ}}{|\overrightarrow{PQ}|}\right) = \frac{2}{\sqrt{11}}\langle 3,-1,1\rangle$, AND $\vec{a}(t) = \vec{a} = \langle 3,-1,1\rangle$

1) $\vec{v}(t) = \int \vec{a}(t)\,dt = \int \vec{a}\,dt = \vec{a}t + \vec{C} = \vec{a}t + \vec{v}(0) = \vec{a}t + \frac{2}{\sqrt{11}}\vec{a}$

2) $\vec{r}(t) = \int \vec{v}(t)\,dt = \int \left(t + \frac{2}{\sqrt{11}}\right)\vec{a}\,dt = \left(\frac{t^2}{2} + \frac{2}{\sqrt{11}}t\right)\vec{a} + \vec{D}$

$= \left(\frac{t^2}{2} + \frac{2}{\sqrt{11}}t\right)\vec{a} + \vec{r}(0)$

$= \boxed{\left(\frac{t^2}{2} + \frac{2}{\sqrt{11}}t\right)\langle 3,-1,1\rangle + \langle 1,2,3\rangle}$

$= \boxed{\left\langle \frac{3}{2}t^2 + \frac{6}{\sqrt{11}}t + 1,\ -\frac{t^2}{2} - \frac{2}{\sqrt{11}}t + 2,\ \frac{t^2}{2} + \frac{2}{\sqrt{11}}t + 3\right\rangle}$

MATH 21C SOLUTIONS – SEC. 14.7 AND 13.2

14.7 – (50) WE WANT TO MINIMIZE THE DISTANCE d FROM A POINT (x, y, z) ON THE PARABOLOID $z = x^2 + y^2 + 10$ TO THE PLANE $x + 2y - z = 0$.

SINCE $d = \dfrac{|x + 2y - z|}{\sqrt{6}} = \dfrac{|z - x - 2y|}{\sqrt{6}} = \dfrac{|x^2 + y^2 + 10 - x - 2y|}{\sqrt{6}}$

$= \dfrac{1}{\sqrt{6}}\left(x^2 + y^2 - x - 2y + 10\right)$ $\left[\text{SINCE } x^2 + y^2 - x - 2y + 10 = (x - \tfrac{1}{2})^2 + (y-1)^2 + \tfrac{35}{4} \geq 0\right]$,

WE CAN MINIMIZE $f(x, y) = x^2 + y^2 - x - 2y + 10$:

$f_x = 2x - 1 = 0$ AND $f_y = 2y - 2 = 0$ GIVES $x = 1/2$, $y = 1$; AND THEN d HAS A MIN, SINCE $f_{xx} = 2$, $f_{xy} = 0$, $f_{yy} = 2$, AND $D = 2(2) - 0^2 = 4$;

SO $D > 0$ AND $f_{xx} > 0$. THEREFORE $\boxed{(1/2, 1, 45/4)}$ IS THE CLOSEST POINT

(SINCE $z = (\tfrac{1}{2})^2 + 1^2 + 10 = \tfrac{45}{4}$).

OR MINIMIZE $d = \dfrac{|x + 2y - z|}{\sqrt{6}} = \dfrac{1}{\sqrt{6}}(z - x - 2y)$ (SINCE $x + 2y - z < 0$ FOR POINTS ON THE PARABOLOID, AS SHOWN ABOVE)

SUBJECT TO THE CONSTRAINT $z = x^2 + y^2 + 10$ OR $\underbrace{x^2 + y^2 - z}_{g(x,y,z)} = -10$:

$-\frac{1}{\sqrt{6}} = \lambda \cdot 2x$ SO $\lambda = -\frac{1}{\sqrt{6}}$: $-\frac{1}{\sqrt{6}} = -\frac{1}{\sqrt{6}} \cdot 2x$ SO $\boxed{x = 1/2}$

$-\frac{2}{\sqrt{6}} = \lambda \cdot 2y$ $\qquad -\frac{2}{\sqrt{6}} = -\frac{1}{\sqrt{6}} \cdot 2y$ SO $\boxed{y = 1}$

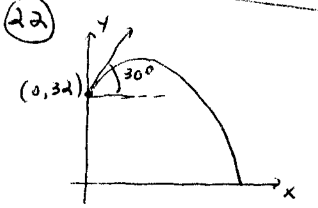
$\frac{1}{\sqrt{6}} = \lambda(-1)$ $\qquad$ THEN $\boxed{z = \frac{45}{4}} \longleftarrow (\tfrac{1}{2})^2 + 1^2 + 10$

OR BY MOVING THE PLANE PARALLEL TO ITSELF UNTIL IT FIRST TOUCHES THE PARABOLOID, WE SEE THAT THE TANGENT PLANE AT THE CLOSEST POINT MUST BE PARALLEL TO THE GIVEN PLANE, SO THEIR NORMAL VECTORS MUST BE PARALLEL:

$\vec{n} = \langle f_x, f_y, -1 \rangle = \langle 2x, 2y, -1 \rangle = k\langle 1, 2, -1 \rangle$ GIVES

$2x = k$, $2y = 2k$, AND $k = 1$; SO $\boxed{x = 1/2}$, $\boxed{y = 1}$, $\boxed{z = \frac{45}{4}}$

13.2 – (22)



TAKING $x_0 = 0$ AND $y_0 = 32$, WE HAVE

$x = (v_0 \cos\alpha)t + x_0 = (32 \cdot \frac{\sqrt{3}}{2})t + 0 = 16\sqrt{3}\,t$ AND

$y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + y_0$

$= (32 \cdot \frac{1}{2})t - \frac{1}{2} \cdot 32t^2 + 32 = -16t^2 + 16t + 32$

THEN $y = 0$ WHEN $-16t^2 + 16t + 32 = 0$, $t^2 - t - 2 = 0$, $(t-2)(t+1) = 0$

$\boxed{t = 2 \text{ sec}}$ OR $\cancel{t = -1}$

AFTER 2 SEC, IT HAS MOVED $x = 16\sqrt{3}(2) = \boxed{32\sqrt{3} \text{ FT}}$ HORIZONTALLY.

(25) $x = (400\cos\alpha)t$, $y = (400\sin\alpha)t - \frac{1}{2}(9.8)t^2$

SO $y = 0$ WHEN $t = \dfrac{400\sin\alpha}{4.9}$, AND THEN

$x = (400\cos\alpha)\left(\dfrac{400\sin\alpha}{4.9}\right) = \dfrac{(400)^2 \cdot \sin\alpha\cos\alpha}{4.9} = \dfrac{(400)^2 \overbrace{\sin 2\alpha}^{2\sin\alpha\cos\alpha}}{9.8}$ *

SO $x = 16{,}000$ GIVES $\dfrac{(400)^2 \sin 2\alpha}{9.8} = 16{,}000$, $\sin 2\alpha = 9.8(.1) = .98$

SO $2\alpha \approx 78.52°$ OR $2\alpha \approx 180° - 78.52° \approx 101.48°$

$\alpha \approx \boxed{39.3°}$ $\qquad \alpha \approx \boxed{50.7°}$

* (OR USE THE FORMULA $R = \dfrac{v_0^2}{g}\sin 2\alpha$ FOR THE RANGE.)