(12) $\sum_{n=1}^{\infty} e^{-n} = \sum_{n=1}^{\infty} \frac{1}{e^n} = \sum_{n=1}^{\infty} \left(\frac{1}{e}\right)^n$ $\boxed{\text{CONVERGES}}$ SINCE IT IS A GEOMETRIC SERIES WITH $|r|<1$.

(13) $\sum_{n=1}^{\infty} \frac{n}{n+1}$ $\boxed{\text{DIVERGES}}$ BY THE DIVERGENCE TEST SINCE $\lim_{n\to\infty} \frac{n}{n+1} = 1 \neq 0$.

(14) $\sum_{n=1}^{\infty} \frac{5}{n+1}$ $\boxed{\text{DIVERGES}}$ SINCE IT'S A MULTIPLE OF THE HARMONIC SERIES WITH 1 TERM DELETED.

(15) $\sum_{n=1}^{\infty} \frac{3}{\sqrt{n}}$ $\boxed{\text{DIVERGES}}$ SINCE IT'S A MULTIPLE OF A P-SERIES WITH $P \leq 1$.

(16) $\sum_{n=1}^{\infty} \frac{-2}{n\sqrt{n}} = \sum_{n=1}^{\infty} \frac{-2}{n^{3/2}}$ $\boxed{\text{CONVERGES}}$ SINCE IT'S A MULTIPLE OF A P-SERIES WITH $P>1$.

(33) $\sum_{n=1}^{\infty} n \sin\frac{1}{n}$ $\boxed{\text{DIVERGES}}$ BY THE DIVERGENCE TEST SINCE

$$\lim_{n\to\infty} n\sin\frac{1}{n} = \lim_{n\to\infty} \frac{\sin\frac{1}{n}}{\frac{1}{n}} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1 \neq 0$$

(IF $x = \frac{1}{n}$, THEN $x \to 0^+$ AS $n\to\infty$)

(38) $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n}$, WHICH DIVERGES (HARMONIC SERIES):

$\lim_{n\to\infty} \frac{n}{n^2+1} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 \neq 0$, SO $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $\boxed{\text{DIVERGES}}$ BY THE LCT.

$\boxed{\text{OR}}$ IF $f(x) = \frac{x}{x^2+1}$, THEN $f'(x) = \frac{(x^2+1)\cdot 1 - x\cdot 2x}{(x^2+1)^2} = \frac{1-x^2}{(x^2+1)^2} = \frac{(1-x)(1+x)}{(x^2+1)^2} < 0$ FOR $x>1$;

SO f IS CONTINUOUS AND DECREASING FOR $x \geq 1$.

SINCE $\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{T\to\infty} \int_1^T \frac{x}{x^2+1}\,dx = \lim_{T\to\infty} \frac{1}{2}\int_1^T \frac{2x}{x^2+1}\,dx$ $\left(\frac{u = x^2+1}{u' = 2x}\right)$

$= \lim_{T\to\infty} \frac{1}{2}\left[\ln(x^2+1)\right]_1^T = \lim_{T\to\infty} \frac{1}{2}\left(\ln(T^2+1) - \ln 2\right) = \infty$,

$\int_1^{\infty} \frac{x}{x^2+1}\,dx$ DIVERGES AND THEREFORE $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ $\boxed{\text{DIVERGES}}$ BY THE INTEGRAL TEST.

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(55) a) $\int_2^{\infty} \frac{dx}{x(\ln x)^p} = \lim_{T\to\infty} \int_2^T \frac{dx}{x(\ln x)^p}$ $\quad \underline{u = \ln x}$, $du = \frac{1}{x}\,dx$

1) IF $\underline{p=1}$, $\lim_{T\to\infty} \int_2^T \frac{dx}{x\ln x} = \lim_{T\to\infty} \int_{\ln 2}^{\ln T} \frac{1}{u}\,du = \lim_{T\to\infty} \left[\ln u\right]_{\ln 2}^{\ln T} = \lim_{T\to\infty} \left(\ln(\ln T) - \ln(\ln 2)\right) = \infty$.

2) IF $\underline{p\neq 1}$, $\lim_{T\to\infty} \int_2^T \frac{dx}{x(\ln x)^p} = \lim_{T\to\infty} \int_{\ln 2}^{\ln T} \frac{1}{u^p}\,du = \lim_{T\to\infty} \int_{\ln 2}^{\ln T} u^{-p}\,du = \lim_{T\to\infty} \left[\frac{u^{1-p}}{1-p}\right]_{\ln 2}^{\ln T}$

$$= \lim_{T\to\infty} \frac{1}{1-p}\left((\ln T)^{1-p} - (\ln 2)^{1-p}\right) = \begin{cases} \infty, & \text{IF } 0<p<1 \\ \frac{1}{p-1}(\ln 2)^{1-p}, & \text{IF } p>1 \end{cases}$$

THEREFORE $\int_2^{\infty} \frac{dx}{x(\ln x)^p}$ CONVERGES IFF $p>1$.

b) SINCE $f(x) = \frac{1}{x(\ln x)^p}$ IS CONTINUOUS AND DECREASING FOR $x \geq 2$,

$\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ $\underline{\text{CONVERGES}}$ IF $p>1$ AND $\underline{\text{DIVERGES}}$ FOR $0<p\leq 1$ BY THE INTEGRAL TEST AND PART a).

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(2) $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n^3}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P>1$:

$$\frac{n-1}{n^4+2} < \frac{n}{n^4+2} < \frac{n}{n^4} = \frac{1}{n^3} \quad \text{FOR } n \geq 1,$$

SO $\sum_{n=1}^{\infty} \frac{n-1}{n^4+2}$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.

(4) $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ COMPARE TO $\sum_{n=2}^{\infty} \frac{1}{n}$, WHICH DIVERGES SINCE IT IS THE HARMONIC SERIES WITH 1 TERM DELETED:

$$\frac{n+2}{n^2-n} > \frac{n}{n^2-n} > \frac{n}{n^2} = \frac{1}{n} \quad \text{FOR } n \geq 2,$$

SO $\sum_{n=2}^{\infty} \frac{n+2}{n^2-n}$ $\boxed{\text{DIVERGES}}$ BY THE COMPARISON TEST.

(6) $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{3^n} = \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$, WHICH CONVERGES SINCE IT IS A GEOMETRIC SERIES WITH $|r|<1$:

$$\frac{1}{n3^n} \leq \frac{1}{3^n} \quad \text{FOR } n \geq 1,$$

SO $\sum_{n=1}^{\infty} \frac{1}{n3^n}$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.

(18) $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n}$, WHICH DIVERGES SINCE IT'S THE HARMONIC SERIES:

$$\lim_{n\to\infty} \frac{3}{n+\sqrt{n}} \div \frac{1}{n} = \lim_{n\to\infty} \frac{3n}{n+\sqrt{n}} = \lim_{n\to\infty} \frac{3}{1+\frac{1}{\sqrt{n}}} = \frac{3}{1+0} = 3 \neq 0,$$

SO $\sum_{n=1}^{\infty} \frac{3}{n+\sqrt{n}}$ $\boxed{\text{DIVERGES}}$ BY THE LIMIT COMPARISON TEST.

(OR USE THE COMPARISON TEST, AND THE FACT THAT
$\frac{3}{n+\sqrt{n}} \geq \frac{1}{n}$ IFF $3n \geq n+\sqrt{n}$ IFF $2n \geq \sqrt{n}$ IFF $2\sqrt{n} \geq 1$ IFF $4n \geq 1$ IFF $n \geq \frac{1}{4}$)

(19) $\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$, WHICH CONVERGES SINCE IT IS A GEOMETRIC SERIES WITH $|r|<1$:

$\underline{0 \leq \sin^2 n \leq 1}$ FOR ALL n, SO

$0 \leq \frac{\sin^2 n}{2^n} \leq \frac{1}{2^n}$ FOR ALL n AND THEREFORE

$\sum_{n=1}^{\infty} \frac{\sin^2 n}{2^n}$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.

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(5) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$ COMPARE TO $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P > 1$!

$0 \le \cos^2 n \le 1$ FOR ALL n, SO $0 \le \dfrac{\cos^2 n}{n^{3/2}} \le \dfrac{1}{n^{3/2}}$ FOR ALL n

AND THEREFORE $\displaystyle\sum_{n=1}^{\infty} \frac{\cos^2 n}{n^{3/2}}$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.

(9) $\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$ COMPARE TO $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P > 1$!

$\displaystyle\lim_{n\to\infty} \frac{n-2}{n^3-n^2+3} \div \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^3 - 2n^2}{n^3 - n^2 + 3} = 1 \neq \infty$, SO

$\displaystyle\sum_{n=1}^{\infty} \frac{n-2}{n^3 - n^2 + 3}$ $\boxed{\text{CONVERGES}}$ BY THE LIMIT COMPARISON TEST.

(OR USE THE COMPARISON TEST AND THE FACT THAT

$\dfrac{n-2}{n^3-n^2+3} \le \dfrac{1}{n^2}$ IFF $n^3 - 2n^2 \le n^3 - n^2 + 3$ IFF $-3 \le n^2$, WHICH IS TRUE FOR ALL n)

(11) $\displaystyle\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$ COMPARE TO $\displaystyle\sum_{n=2}^{\infty} \frac{n^2}{n^3} = \sum_{n=2}^{\infty} \frac{1}{n}$, WHICH DIVERGES SINCE IT'S THE HARMONIC SERIES WITH ONE TERM DELETED!

$\displaystyle\lim_{n\to\infty} \frac{n(n+1)}{(n^2+1)(n-1)} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^2(n+1)}{(n^2+1)(n-1)} = \lim_{n\to\infty} \frac{n^3+n^2}{n^3 - n^2 + n - 1} = 1 > 0$,

SO $\displaystyle\sum_{n=2}^{\infty} \frac{n(n+1)}{(n^2+1)(n-1)}$ $\boxed{\text{DIVERGES}}$ BY THE LIMIT COMPARISON TEST.

(OR USE THE COMPARISON TEST AND THE FACT THAT

$\dfrac{n(n+1)}{(n^2+1)(n-1)} \ge \dfrac{1}{n}$ IFF $n^2(n+1) \ge (n^2+1)(n-1)$ IFF $n^3 + n^2 \ge n^3 - n^2 + n - 1$

IFF $2n^2 - n \ge -1$ IFF $n(2n-1) \ge -1$, WHICH IS TRUE FOR ALL n)

(21) $\displaystyle\sum_{n=1}^{\infty} \frac{2n}{3n-1}$ $\displaystyle\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{2n}{3n-1} = \frac{2}{3} \neq 0$,

SO THE SERIES $\boxed{\text{DIVERGES}}$ BY THE DIVERGENCE TEST.

(20) $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{2}{n^2}$, WHICH CONVERGES SINCE IT'S A MULTIPLE OF A P-SERIES WITH $P>1$:

$-1 \le \cos n \le 1$ FOR ALL n, SO

$0 \le 1+\cos n \le 2$ FOR ALL n AND $0 \le \frac{1+\cos n}{n^2} \le \frac{2}{n^2}$ FOR ALL n.

THEREFORE $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.*

(25) $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$ COMPARE TO $\sum_{n=1}^{\infty} \left(\frac{1}{3}\right)^n$, WHICH CONVERGES SINCE IT IS A GEOMETRIC SERIES WITH $-1<r<1$:

$\frac{n}{3n+1} < \frac{1}{3}$ FOR ALL n, SO $\left(\frac{n}{3n+1}\right)^n < \left(\frac{1}{3}\right)^n$ FOR ALL n AND

THEREFORE $\sum_{n=1}^{\infty} \left(\frac{n}{3n+1}\right)^n$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.

(27) $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ COMPARE TO $\sum_{n=3}^{\infty} \frac{1}{n}$, WHICH DIVERGES SINCE IT'S THE HARMONIC SERIES WITH 2 TERMS DELETED:

$\ln n < n$ FOR ALL n, SO $\ln(\ln n) < \ln n < n$ FOR $n \ge 2$ AND

THEREFORE $\frac{1}{\ln(\ln n)} > \frac{1}{n}$ FOR $n \ge 3$ (SINCE $\ln(\ln n) > 0$ FOR $n \ge 3$)

THUS $\sum_{n=3}^{\infty} \frac{1}{\ln(\ln n)}$ $\boxed{\text{DIVERGES}}$ BY THE COMPARISON TEST.

(OR USE THE LIMIT COMPARISON TEST, AND THE FACT THAT

$$\lim_{n\to\infty} \frac{1}{\ln(\ln n)} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n}{\ln(\ln n)} = \lim_{x\to\infty} \frac{x}{\ln(\ln x)}$$

$$\underset{\left(\frac{\infty}{\infty}\right)}{=} \lim_{x\to\infty} \frac{1}{\frac{1/x}{\ln x}} = \lim_{x\to\infty} \frac{1}{\frac{1}{x\ln x}} = \lim_{x\to\infty} x\ln x = \infty \ne 0 \Big)$$

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*REMARK NOTICE THAT USING THE LCT TO COMPARE $\sum_{n=1}^{\infty} \frac{1+\cos n}{n^2}$ TO $\sum_{n=1}^{\infty} \frac{2}{n^2}$ IN #20 WOULD NOT WORK,

SINCE $\lim_{n\to\infty} (1+\cos n)$ DOES NOT EXIST.

(32) $$\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1} = \sum_{n=3}^{\infty} \frac{\ln n}{n}$$

COMPARE TO $\sum_{n=3}^{\infty} \frac{1}{n}$, WHICH DIVERGES SINCE IT'S THE HARMONIC SERIES WITH 2 TERMS DELETED:

$$\frac{\ln n}{n} \geq \frac{1}{n} \quad \text{IFF} \quad \ln n \geq 1 \quad \text{IFF} \quad n \geq e,$$

SO $\sum_{n=2}^{\infty} \frac{\ln(n+1)}{n+1} = \sum_{n=3}^{\infty} \frac{\ln n}{n}$ $\boxed{\text{DIVERGES}}$ BY THE COMPARISON TEST.

(34) $$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$$

COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P > 1$:

$$\frac{\sqrt{n}}{n^2+1} < \frac{\sqrt{n}}{n^2} = \frac{1}{n^{3/2}} \quad \text{FOR } n \geq 1, \text{ SO}$$

$\sum_{n=1}^{\infty} \frac{\sqrt{n}}{n^2+1}$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.

(40) $$\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$$

COMPARE TO $\sum_{n=1}^{\infty} \frac{3^n}{4^n} = \sum_{n=1}^{\infty} \left(\frac{3}{4}\right)^n$, WHICH CONVERGES SINCE IT IS A GEOMETRIC SERIES WITH $-1 < r < 1$:

$$\frac{2^n+3^n}{3^n+4^n} \leq \frac{3^n}{4^n} \quad \text{IFF} \quad (2^n+3^n)4^n \leq (3^n+4^n)3^n$$

IFF $8^n + 12^n \leq 9^n + 12^n$ IFF $8^n \leq 9^n$,

WHICH IS TRUE FOR ALL n;

SO $\sum_{n=1}^{\infty} \frac{2^n+3^n}{3^n+4^n}$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.

(OR USE THE LIMIT COMPARISON TEST, AND THE FACT THAT

$$\lim_{n\to\infty} \frac{2^n+3^n}{3^n+4^n} \div \frac{3^n}{4^n} = \lim_{n\to\infty} \frac{\frac{2^n}{3^n} + \frac{3^n}{3^n}}{\frac{3^n}{4^n} + \frac{4^n}{4^n}} = \lim_{n\to\infty} \frac{\left(\frac{2}{3}\right)^n + 1}{\left(\frac{3}{4}\right)^n + 1}$$

$$= \frac{0+1}{0+1} = 1 \neq \infty)$$
