(16) $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n^2}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P>1$:

$$\lim_{n\to\infty} \ln\left(1+\frac{1}{n^2}\right) \div \frac{1}{n^2} = \lim_{x\to\infty} \frac{\ln\left(1+\frac{1}{x^2}\right)}{\frac{1}{x^2}} = \lim_{u\to 0^+} \frac{\ln(1+u)}{u} = \lim_{u\to 0^+} \frac{\frac{1}{1+u}}{1} = 1 \neq \infty,$$

SO $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n^2}\right)$ $\boxed{\text{CONVERGES}}$ BY THE LIMIT COMPARISON TEST.

(OR USE THE COMPARISON TEST AND THE FACT THAT $\ln(1+x) < x$ FOR $x>0$, SINCE $f(x) = x - \ln(1+x)$ SATISFIES $f(0)=0$ AND $f'(x)>0$ FOR $x>0$)

(28) $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n^2}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P>1$:

$\ln n < \sqrt{n}$ FOR $n \geq N$ (FOR SOME N)*, SO

$(\ln n)^2 < n$ AND $\frac{(\ln n)^2}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}$ FOR $n \geq N$.

THEREFORE $\sum_{n=1}^{\infty} \frac{(\ln n)^2}{n^3}$ $\boxed{\text{CONVERGES}}$ BY THE COMPARISON TEST.

(OR USE THE LCT BY SHOWING THAT $\lim_{n\to\infty} \frac{(\ln n)^2}{n^3} \div \frac{1}{n^2} = \lim_{x\to\infty} \frac{(\ln x)^2}{x} = 0 \neq \infty$)

* THIS IS TRUE FOR $n \geq 1$, SINCE $f(x) = \sqrt{x} - \ln x$ HAS AN ABS. MIN. GIVEN BY $f(4) = 2(1-\ln 2) > 0$ SINCE $2 < e \Rightarrow \ln 2 < 1$.

(27) $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ COMPARE TO $\sum_{n=2}^{\infty} \frac{1}{n}$, WHICH DIVERGES SINCE IT IS THE HARMONIC SERIES (WITH 1 TERM DELETED):

$\ln n < \sqrt{n}$ FOR $n \geq 1$, SO $\sqrt{n}\ln n < n$ FOR $n \geq 1$ AND THEREFORE

$\frac{1}{\sqrt{n}\ln n} > \frac{1}{n}$ FOR $n \geq 2$.

THEREFORE $\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}\ln n}$ $\boxed{\text{DIVERGES}}$ BY THE COMPARISON TEST.

(OR USE THE LCT BY SHOWING THAT $\lim_{n\to\infty} \frac{1}{\sqrt{n}\ln n} \div \frac{1}{n} = \lim_{n\to\infty} \frac{\sqrt{n}}{\ln n} = \lim_{x\to\infty} \frac{\sqrt{x}}{\ln x} = \infty \neq 0$)

(41) $$\sum_{n=1}^{\infty} \frac{2^n - n}{n2^n} = \sum_{n=1}^{\infty} \left(\frac{2^n}{n2^n} - \frac{n}{n2^n}\right) = \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{2^n}\right)$$

1) $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES SINCE IT IS THE HARMONIC SERIES, AND

2) $\sum_{n=1}^{\infty} \frac{1}{2^n} = \sum_{n=1}^{\infty} \left(\frac{1}{2}\right)^n$ CONVERGES SINCE IT IS A GEOMETRIC SERIES WITH $|r|<1$,

SO $\sum_{n=1}^{\infty} \frac{2^n - n}{n2^n}$ $\boxed{\text{DIVERGES}}$ (SINCE ONE SERIES CONVERGES AND THE OTHER SERIES DIVERGES)

(OR USE THE LCT TO COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n}$, WHICH DIVERGES SINCE IT'S THE HARMONIC SERIES:

$$\lim_{n\to\infty} \frac{2^n - n}{n2^n} \div \frac{1}{n} = \lim_{n\to\infty} \frac{2^n - n}{2^n} = \lim_{n\to\infty} \left(1 - \frac{n}{2^n}\right) = 1 - \lim_{x\to\infty} \underbrace{\frac{x}{2^x}}_{\left(\frac{\infty}{\infty}\right)} = 1 - \lim_{x\to\infty} \frac{1}{2^x \ln 2}$$

$$= 1 - 0 = 1 \neq 0,$$

SO $\sum_{n=1}^{\infty} \frac{2^n - n}{n2^n}$ $\boxed{\text{DIVERGES}}$ BY THE LCT)

SOLUTIONS – 10.4 AND 10.5

10.4 – 33) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ COMPARE TO $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^2}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $p > 1$ (WITH 1 TERM DELETED):

$$\lim_{n\to\infty} \frac{1}{n\sqrt{n^2-1}} \div \frac{1}{n^2} = \lim_{n\to\infty} \frac{n^2}{n\sqrt{n^2-1}} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2-1}}$$

$$= \lim_{n\to\infty} \frac{n}{\sqrt{n^2}\sqrt{1-\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{\cancel{n}}{\cancel{n}\sqrt{1-\frac{1}{n^2}}} = \lim_{n\to\infty} \frac{1}{\sqrt{1-\frac{1}{n^2}}} = 1 \neq \infty,$$

SO $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ $\boxed{\text{CONVERGES}}$ BY THE LIMIT COMPARISON TEST.

55) a) IF $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, THEN $\underline{\frac{a_n}{b_n} \le 1}$ FOR $n \ge N$ (FOR SOME INTEGER N),

THEN $\underline{a_n \le b_n}$ FOR $n \ge N$, SO $\displaystyle\sum_{n=1}^{\infty} a_n$ CONVERGES BY THE COMPARISON TEST SINCE $\displaystyle\sum_{n=1}^{\infty} b_n$ CONVERGES.

b) IF $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \infty$, THEN $\underline{\frac{a_n}{b_n} \ge 1}$ FOR $n \ge N$ (FOR SOME INTEGER N),

THEN $\underline{a_n \ge b_n}$ FOR $n \ge N$, SO $\displaystyle\sum_{n=1}^{\infty} a_n$ DIVERGES BY THE COMPARISON TEST SINCE $\displaystyle\sum_{n=1}^{\infty} b_n$ DIVERGES.

10.5 – 15) $\displaystyle\sum_{n=1}^{\infty} \left(1-\frac{1}{n}\right)^{n^2}$

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\left(1-\frac{1}{n}\right)^{n^2}\right)^{1/n} = \lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n$$

$$= \lim_{n\to\infty} \left(1+\frac{-1}{n}\right)^n = e^{-1} = \frac{1}{e} < 1,$$

SO THE SERIES $\boxed{\text{CONVERGES}}$ BY THE ROOT TEST.

29) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$

1) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES SINCE IT IS THE HARMONIC SERIES, AND

2) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES SINCE IT IS A P-SERIES WITH $p > 1$;

SO $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right)$ $\boxed{\text{DIVERGES}}$.

(OR COMPARE $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n^2}\right) = \sum_{n=1}^{\infty} \frac{n-1}{n^2}$ TO $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ USING THE LCT.)

Sec. 10.5 – SOLUTIONS

(2) $\sum_{n=1}^{\infty} \frac{n+2}{3^n}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n+3}{3^{n+1}} \cdot \frac{\cancel{3^n}}{n+2} = \lim_{n\to\infty} \frac{1}{3}\cdot\frac{n+3}{n+2} = \frac{1}{3}\cdot 1 = \frac{1}{3} < 1,$$

so the series $\boxed{\text{CONVERGES}}$ by the Ratio Test.

(22) $\sum_{n=1}^{\infty} \left(\frac{n-2}{n}\right)^n$

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(\frac{n-2}{n}\right)^n = \lim_{n\to\infty} \left(1 + \frac{-2}{n}\right)^n = e^{-2} \neq 0,$$

so the series $\boxed{\text{DIVERGES}}$ by the Divergence Test.

(27) $\sum_{n=1}^{\infty} \frac{\ln n}{n^3}$

Compare to $\sum_{n=1}^{\infty} \frac{1}{n^2}$, which converges since it's a p-series with $p > 1$!

$\underline{\ln n < n}$ for all n, so $\underline{\frac{\ln n}{n^3} < \frac{n}{n^3} = \frac{1}{n^2}}$ for all n; and therefore

the series $\boxed{\text{CONVERGES}}$ by the Comparison Test.

(or use the LCT instead)

(28) $\sum_{n=1}^{\infty} \frac{(\ln n)^n}{n^n}$

$$\rho = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \underset{\left(\frac{\infty}{\infty}\right)}{\frac{\ln x}{x}} = \lim_{x\to\infty} \frac{1/x}{1} = 0 < 1,$$

so the series $\boxed{\text{CONVERGES}}$ by the Root Test.

(31) $\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!}$$

(since $(2(n+1)+1)! = (2n+3)!$)

$$= \lim_{n\to\infty} \frac{n+1}{(2n+3)(2n+2)} = \lim_{n\to\infty} \frac{1}{2(2n+3)} = 0 < 1,$$

so the series $\boxed{\text{CONVERGES}}$ by the Ratio Test.

(Notice that $\underline{\frac{(2n+1)!}{(2n+3)!}} = \frac{\cancel{(2n+1)(2n)(2n-1)\cdots 1}}{(2n+3)(2n+2)\cancel{(2n+1)(2n)\cdots 1}} = \underline{\frac{1}{(2n+3)(2n+2)}}$)

(32) $\sum_{n=1}^{\infty} \frac{n \ln n}{2^n}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)\ln(n+1)}{2^{n+1}} \cdot \frac{\cancel{2^n}}{n \ln n}$$

$$= \lim_{n\to\infty} \frac{1}{2}\cdot\frac{n+1}{n}\cdot\frac{\ln(n+1)}{\ln n} = \frac{1}{2}\cdot 1\cdot \lim_{x\to\infty} \underset{\left(\frac{\infty}{\infty}\right)}{\frac{\ln(x+1)}{\ln x}} = \frac{1}{2}\cdot\lim_{x\to\infty} \frac{\frac{1}{x+1}}{\frac{1}{x}}$$

$$= \frac{1}{2}\cdot\lim_{x\to\infty} \frac{x}{x+1} = \frac{1}{2}\cdot 1 = \frac{1}{2} < 1,$$

so the series $\boxed{\text{CONVERGES}}$ by the Ratio Test.

(or use the Root Test and the facts that

$\underline{\lim_{n\to\infty} n^{1/n} = 1}$ by L'Hospital's Rule (as we showed in Sec. 10.1) and

$\underline{\lim_{n\to\infty} (\ln n)^{1/n} = 1}$ by L'Hospital's Rule:

$$\lim_{n\to\infty} (\ln n)^{1/n} = \lim_{n\to\infty} e^{\ln((\ln n)^{1/n})} = \lim_{n\to\infty} e^{\frac{1}{n}\ln(\ln n)} = e^{\lim_{n\to\infty} \frac{\ln(\ln n)}{n}}$$

$$= e^{\lim_{x\to\infty} \frac{\ln(\ln x)}{x}} \underset{\left(\frac{\infty}{\infty}\right)}{=} e^{\lim_{x\to\infty} \frac{\frac{1}{x}\cdot\frac{1}{\ln x}}{1}} = e^{\lim_{x\to\infty} \frac{1}{x\ln x}} = e^0 = 1.$$)