MATH 21C

SOLUTIONS - SEC. 10.7 AND 10.8

10.7 - (12) $\displaystyle\sum_{n=0}^{\infty} \frac{n(x+3)^n}{5^n}$

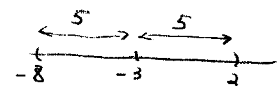
1) $\displaystyle \ell = \lim_{n\to\infty} |u_n|^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}|x+3|}{5} = 1\cdot\frac{|x+3|}{5} = \frac{|x+3|}{5}$

2) $\underline{\ell < 1}$ IFF $\dfrac{|x+3|}{5} < 1$ IFF $|x+3| < 5$ IFF $-5 < x+3 < 5$ IFF $\underline{-8 < x < 2}$

3) a) IF $\underline{x=2}$, WE GET $\displaystyle\sum_{n=0}^{\infty} \frac{n\cdot 5^n}{5^n} = \sum_{n=0}^{\infty} n$, WHICH DIVERGES BY THE DIVERGENCE TEST SINCE $\displaystyle\lim_{n\to\infty} n = \infty \neq 0$,

b) IF $\underline{x=-8}$, WE GET $\displaystyle\sum_{n=0}^{\infty} \frac{n(-5)^n}{5^n} = \sum_{n=0}^{\infty} (-1)^n n$, WHICH DIVERGES FOR THE SAME REASON

INTERVAL OF CONV. : $\boxed{(-8,2)}$ RADIUS OF CONV. $=\boxed{5}$



(27a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x+2)^n}{n\,2^n}$

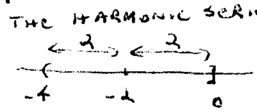
1) $\displaystyle \ell = \lim_{n\to\infty} |u_n|^{1/n} = \lim_{n\to\infty} \frac{|x+2|}{n^{1/n}\cdot 2} = \frac{|x+2|}{1\cdot 2} = \frac{|x+2|}{2}$

2) $\underline{\ell < 1}$ IFF $\dfrac{|x+2|}{2} < 1$ IFF $|x+2| < 2$ IFF $-2 < x+2 < 2$ IFF $\underline{-4 < x < 0}$

3) a) IF $\underline{x=0}$, WE GET $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1}\cdot\frac{1}{n}$, WHICH CONVERGES BY THE AST SINCE i) $\displaystyle\lim_{n\to\infty}\frac{1}{n} = 0$ AND ii) $\dfrac{1}{n} \geq \dfrac{1}{n+1}$.

b) IF $\underline{x=-4}$, WE GET $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-2)^n}{n\,2^n} = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(-1)^n\cancel{2^n}}{n\cancel{2^n}} = -\sum_{n=1}^{\infty}\frac{1}{n}$, WHICH DIVERGES SINCE IT'S A MULTIPLE OF THE HARMONIC SERIES

INTERVAL OF CONV. : $\boxed{(-4,0]}$ RADIUS OF CONV. $=\boxed{2}$



10.8 - (3) $f(x) = \ln x$, $a = 1$

$f'(x) = \frac{1}{x} = x^{-1}$

$f''(x) = -x^{-2}$

$f'''(x) = 2x^{-3}$

n	$f^{(n)}(a)$	$\dfrac{f^{(n)}(a)}{n!}$
0	0	0
1	1	1
2	-1	-1/2
3	2	1/3

$$\boxed{\begin{aligned} P_0(x) &= 0 \\ P_1(x) &= 1(x-1) = (x-1) \\ P_2(x) &= (x-1) - \tfrac{1}{2}(x-1)^2 \\ P_3(x) &= (x-1) - \tfrac{1}{2}(x-1)^2 + \tfrac{1}{3}(x-1)^3 \end{aligned}}$$

(9) $f(x) = \sqrt{x}$, $a = 4$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$

$f'''(x) = \frac{3}{8}x^{-5/2}$

n	$f^{(n)}(a)$	$\dfrac{f^{(n)}(a)}{n!}$
0	2	2
1	$\frac{1}{2}\cdot\frac{1}{2}$	1/4
2	$-\frac{1}{4}\cdot\frac{1}{8}$	-1/64
3	$\frac{3}{8}\cdot\frac{1}{32}$	1/512

$$\boxed{\begin{aligned} P_0(x) &= 2 \\ P_1(x) &= 2 + \tfrac{1}{4}(x-4) \\ P_2(x) &= 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{64}(x-4)^2 \\ P_3(x) &= 2 + \tfrac{1}{4}(x-4) - \tfrac{1}{64}(x-4)^2 + \tfrac{1}{512}(x-4)^3 \end{aligned}}$$

(11) $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$

SO $\underline{e^{-x}} = 1 + (-x) + \dfrac{(-x)^2}{2!} + \dfrac{(-x)^3}{3!} + \cdots = \boxed{1 - x + \dfrac{x^2}{2!} - \dfrac{x^3}{3!} + \cdots} = \boxed{\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}}$

(13) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \boxed{1 - x + x^2 - x^3 + \cdots} = \boxed{\displaystyle\sum_{n=0}^{\infty}(-1)^n x^n}$

$\left(\dfrac{a}{1-r} \text{ WITH } \underline{a=1},\ \underline{r=-x}\right)$

(15) $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \dfrac{x^7}{7!} + \cdots$, SO

$\sin 3x = (3x) - \dfrac{(3x)^3}{3!} + \dfrac{(3x)^5}{5!} - \dfrac{(3x)^7}{7!} + \cdots$

$= \boxed{3x - \dfrac{3^3x^3}{3!} + \dfrac{3^5x^5}{5!} - \dfrac{3^7x^7}{7!} + \cdots} = \boxed{\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{3^{2n+1}x^{2n+1}}{(2n+1)!}}$

SOLUTIONS - SEC. 10.7 AND 10.8

10.7 - 50b

$$g(x) = \frac{3}{x-2} = \frac{-3}{2-x} = \frac{-3/2}{1-x/2} \leftarrow \left(\frac{a}{1-r}, \text{ WITH } a = -\frac{3}{2} \text{ AND } r = \frac{x}{2}\right)$$

$$= \boxed{-\frac{3}{2} - \frac{3x}{4} - \frac{3x^2}{8} - \frac{3x^3}{16} - \cdots} = \boxed{\sum_{n=0}^{\infty} \frac{-3}{2^{n+1}} x^n} \leftarrow (ar^n)$$

THE SERIES CONVERGES FOR $|r|<1$: $\left|\frac{x}{2}\right| < 1$ IFF $|x|<2$ IFF $-2<x<2$, SO $\boxed{(-2,2)}$ IS THE INTERVAL OF CONVERGENCE.

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$$g(x) = \frac{3}{x-2} = \frac{3}{3+(x-5)} = \frac{1}{1+\frac{x-5}{3}} = \frac{1}{1-\left(-\frac{x-5}{3}\right)} \leftarrow \left(\frac{a}{1-r}, \text{ WITH } a=1 \text{ AND } r = -\frac{x-5}{3}\right)$$

$$= \boxed{1 - \frac{x-5}{3} + \frac{(x-5)^2}{9} - \frac{(x-5)^3}{27} + \cdots} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{(x-5)^n}{3^n}} \leftarrow (ar^n)$$

THE SERIES CONVERGES FOR $|r|<1$: $\left|-\frac{x-5}{3}\right|<1$ IFF $\left|\frac{x-5}{3}\right|<1$ IFF $|x-5|<3$ IFF $2<x<8$, SO $\boxed{(2,8)}$ IS THE INTERVAL OF CONVERGENCE.

10.8 - 32

$f(x) = \sqrt{x+1}$, $a=0$

$f(x) = (x+1)^{1/2}$

$f'(x) = \frac{1}{2}(x+1)^{-1/2}$

$f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)(x+1)^{-3/2}$

$f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)(x+1)^{-5/2}$

$f^{(4)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)(x+1)^{-7/2}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	1
1	$1/2$	$1/2$
2	$\frac{1}{2}\left(-\frac{1}{2}\right)$	$-1/8$
3	$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$	$1/16$
4	$\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)$	$-5/128$

$$\sqrt{x+1} = \boxed{1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots}$$

$$= \boxed{1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1\cdot 3\cdot 5\cdot 7\cdots(2n-3)}{2^n n!} x^n}$$

(USING THE CONVENTION THAT THE NUMERATOR IS 1 WHEN $n=1$, SINCE IT IS AN "EMPTY PRODUCT")

$$= \boxed{1 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-2)!}{2^{2n-1}(n-1)!\,n!} x^n}$$

(MULTIPLYING ON THE TOP AND BOTTOM BY $2\cdot 4\cdot 6\cdots(2n-2) = 2^{n-1}(n-1)!$)

(OR USE THE BINOMIAL SERIES FORMULA WITH $m = \frac{1}{2}$)

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$$(1-x+x^2)e^x = (1-x+x^2)\left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots\right)$$

$$= \boxed{1 + \frac{1}{2}x^2 + \frac{2}{3}x^3 + \frac{3}{8}x^4 + \cdots}$$

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$$(\sin x)(\ln(1+x)) = \left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)\left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)$$

$$= \boxed{x^2 - \frac{x^3}{2} + \frac{x^4}{6} - \frac{x^5}{6} + \cdots}$$

(1) $f(x) = e^{2x}$, $a = 0$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

$$e^{2x} = 1 + 2x + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \cdots$$

$$= 1 + 2x + 2x^2 + \frac{4}{3}x^3 + \cdots$$

$$\boxed{\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= 1 + 2x \\ P_2(x) &= 1 + 2x + 2x^2 \\ P_3(x) &= 1 + 2x + 2x^2 + \tfrac{4}{3}x^3 \end{aligned}}$$

(17) $\cosh x = \dfrac{e^x + e^{-x}}{2}$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \frac{x^6}{6!} + \cdots \quad \text{AND}$$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \frac{x^5}{5!} + \frac{x^6}{6!} - \cdots,$$ SO ADDING THESE SERIES GIVES

$$e^x + e^{-x} = 2 + 2\cdot\frac{x^2}{2!} + 2\cdot\frac{x^4}{4!} + 2\cdot\frac{x^6}{6!} + \cdots \quad \text{AND}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} = \boxed{1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots} = \boxed{\sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}}$$

(NOTICE THAT THIS RESEMBLES THE MACLAURIN SERIES FOR $\cos x$.)

(25) $f(x) = x^4 + x^2 + 1$, $a = -2$

$f'(x) = 4x^3 + 2x$

$f''(x) = 12x^2 + 2$

$f'''(x) = 24x$

$f^{(4)}(x) = 24$

$f^{(5)}(x) = 0$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	21	21
1	−36	−36
2	50	25
3	−48	−8
4	24	1
5	0	0

$$x^4 + x^2 + 1 = \boxed{21 - 36(x+2) + 25(x+2)^2 - 8(x+2)^3 + (x+2)^4}$$

(27) $f(x) = \dfrac{1}{x^2}$, $a = 1$

$f(x) = x^{-2}$

$f'(x) = -2x^{-3}$

$f''(x) = 6x^{-4}$

$f'''(x) = -24x^{-5}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	1	1
1	−2	−2
2	6	3
3	−24	−4

$$\frac{1}{x^2} = \boxed{1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + \cdots}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n}$$