MATH 21C SOLUTIONS — PINK SHEET (sec. 10.8 – 10.10)

(1) $f(x) = \dfrac{1}{5x-3}$, $a = 1$

$f(x) = (5x-3)^{-1}$

$f'(x) = -(5x-3)^{-2}\cdot 5$

$f''(x) = 2(5x-3)^{-3}\cdot 5^2$

$f'''(x) = -6(5x-3)^{-4}\cdot 5^3$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	$1/2$	$1/2$
1	$-5/2^2$	$-5/4$
2	$2\cdot 5^2/2^3$	$25/8$
3	$-6\cdot 5^3/2^4$	$-125/16$

$$\frac{1}{5x-3} = \boxed{\frac{1}{2} - \frac{5}{4}(x-1) + \frac{25}{8}(x-1)^2 - \frac{125}{16}(x-1)^3 + \cdots}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{5^n}{2^{n+1}} (x-1)^n}$$

$$\left(\underline{\text{OR}} \text{ use } \frac{1}{5x-3} = \frac{1}{5(x-1)+2} = \frac{1/2}{\frac{5}{2}(x-1)+1} = \frac{1/2}{1-\left(-\frac{5}{2}(x-1)\right)}\right)$$

(2) $f(x) = \dfrac{1}{x^2}$, $a = 2$

$f(x) = x^{-2}$

$f'(x) = -2x^{-3}$

$f''(x) = 6x^{-4}$

$f'''(x) = -24x^{-5}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$	
0	$1/2^2$	$1/4$	
1	$-2/2^3$	$-1/4$	$(-2/8)$
2	$6/2^4$	$3/16$	
3	$-24/2^5$	$-1/8$	$(-4/32)$

$$\frac{1}{x^2} = \boxed{\frac{1}{4} - \frac{1}{4}(x-2) + \frac{3}{16}(x-2)^2 - \frac{1}{8}(x-2)^3 + \cdots}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{n+1}{2^{n+2}} (x-2)^n}$$

$$\left(\underline{\text{OR}} \text{ use } \frac{1}{x^2} = -D_x\left(\frac{1}{x}\right) = -D_x\left(\frac{1}{2+(x-2)}\right) = -D_x\left(\frac{1/2}{1+\frac{x-2}{2}}\right) = -D_x\left(\frac{1/2}{1-\left(-\frac{x-2}{2}\right)}\right)\right)$$

(3) $f(x) = \dfrac{1}{(1-x)^3}$, $a = -1$

$f(x) = (1-x)^{-3}$

$f'(x) = -3(1-x)^{-4}(-1)$

$f''(x) = 3\cdot 4(1-x)^{-5}(-1)^2$

$f'''(x) = 3\cdot 4\cdot(-5)(1-x)^{-6}(-1)^3$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$	
0	$1/2^3$	$1/8$	
1	$3/2^4$	$3/16$	
2	$3\cdot 4/2^5$	$3/16$	$\left(\frac{4!/2}{2^5 2!}\right)$
3	$3\cdot 4\cdot 5/2^6$	$5/32$	$\left(\frac{5!/2}{2^6 3!}\right)$

$$\frac{1}{(1-x)^3} = \boxed{\frac{1}{8} + \frac{3}{16}(x+1) + \frac{3}{16}(x+1)^2 + \frac{5}{32}(x+1)^3 + \cdots}$$

$$= \boxed{\sum_{n=0}^{\infty} \frac{(n+2)(n+1)}{2^{n+4}} (x+1)^n} \qquad \left(\text{since } \frac{\frac{(n+2)!}{2}}{2^{n+3} n!} = \frac{(n+2)!}{2^{n+4} n!} = \frac{(n+2)(n+1)}{2^{n+4}}\right)$$

MATH 216 SOLUTIONS – PINK SHEET AND SEC. 10.9

P.S. – (4) $f(x) = \frac{1}{x}$, $a = 4$

a) $f(x) = x^{-1}$

$f'(x) = -x^{-2}$

$f''(x) = 2x^{-3}$

$f'''(x) = -6x^{-4}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$
0	$1/4$	$1/4$
1	$-1/4^2$	$-1/16$
2	$2/4^3$	$1/64$
3	$-6/4^4$	$-1/256$

$$\frac{1}{x} = \boxed{\frac{1}{4} - \frac{1}{16}(x-4) + \frac{1}{64}(x-4)^2 - \frac{1}{256}(x-4)^3 + \cdots} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{n+1}} (x-4)^n}$$

b) $$\frac{1}{x} = \frac{1}{4+(x-4)} = \frac{1/4}{1+\frac{x-4}{4}} = \frac{\frac{1}{4}}{1-\left(-\frac{x-4}{4}\right)} \leftarrow \frac{a}{1-r}, \text{ with } a = \frac{1}{4},\ r = -\frac{x-4}{4}$$

$$= \boxed{\frac{1}{4} - \frac{1}{16}(x-4) + \frac{1}{64}(x-4)^2 - \frac{1}{256}(x-4)^3 + \cdots} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{1}{4^{n+1}} (x-4)^n} \leftarrow (ar^n)$$

10.9 – (17) $\cos^2 x = \frac{1}{2}(1+\cos 2x)$ where $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$ so

$$\cos 2x = 1 - \frac{(2x)^2}{2!} + \frac{(2x)^4}{4!} - \frac{(2x)^6}{6!} + \cdots = 1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \cdots,$$ and

$$\cos^2 x = \frac{1}{2}(1+\cos 2x) = \frac{1}{2}\left[1 + \left(1 - \frac{4x^2}{2!} + \frac{16x^4}{4!} - \frac{64x^6}{6!} + \cdots\right)\right]$$

$$= \boxed{1 - \frac{2x^2}{2!} + \frac{8x^4}{4!} - \frac{32x^6}{6!} + \cdots} = \boxed{1 + \sum_{n=1}^{\infty} (-1)^n \frac{2^{2n-1} x^{2n}}{(2n)!}}$$

(19) $$\frac{x^2}{1-2x} = \boxed{x^2 + 2x^3 + 4x^4 + 8x^5 + \cdots} = \boxed{\sum_{n=0}^{\infty} 2^n x^{n+2}} \leftarrow (ar^n)$$

$(a = x^2,\ r = +2x)$

(21) $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$ $(a = 1,\ r = x)$, so differentiating gives

$$\frac{1}{(1-x)^2} = D_x\left(\frac{1}{1-x}\right) = \boxed{1 + 2x + 3x^2 + 4x^3 + \cdots} = \boxed{\sum_{n=0}^{\infty} (n+1)x^n}$$

(29) $$e^x \sin x = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right)\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots\right)$$

$$= \left(1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cdots\right)\left(x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots\right)$$

$$= \boxed{x + x^2 + \frac{1}{3}x^3 - \frac{1}{30}x^5 + \cdots}$$ $\leftarrow$ (coeff. of x^5 equals $\frac{1}{24} - \frac{1}{12} + \frac{1}{120} = -\frac{1}{30}$)

(30) $$\frac{\ln(1+x)}{1-x} = (\ln(1+x))\left(\frac{1}{1-x}\right)$$

$$= \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots\right)\left(1 + x + x^2 + x^3 + \cdots\right)$$

$$= \boxed{x + \frac{1}{2}x^2 + \frac{5}{6}x^3 + \frac{7}{12}x^4 + \cdots}$$

MATH 21C SOLUTIONS — Sec. 10.10

3 $(1+x)^{-3} = 1 - 3x + \frac{(-3)(-4)}{2!}x^2 + \frac{(-3)(-4)(-5)}{3!}x^3 + \cdots = 1 - 3x + 6x^2 - 10x^3 + \cdots$ so

$(1-x)^{-3} = 1 - 3(-x) + 6(-x)^2 - 10(-x)^3 + \cdots = \boxed{1 + 3x + 6x^2 + 10x^3 + \cdots}$

15 $\int_0^{.6} \sin x^2\,dx$ $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ so

$\sin x^2 = x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots,$ AND

$$\int_0^{.6} \sin x^2\,dx = \int_0^{.6} \left(x^2 - \frac{x^6}{6} + \frac{x^{10}}{120} - \cdots\right) dx = \left[\frac{x^3}{3} - \frac{x^7}{42} + \frac{x^{11}}{1320} - \cdots\right]_0^{.6}$$

$$= \frac{(.6)^3}{3} - \frac{(.6)^7}{42} + \frac{(.6)^{11}}{1320} - \cdots \approx \boxed{\frac{(.6)^3}{3} - \frac{(.6)^7}{42}} \text{ WITH } |E| < \frac{(.6)^{11}}{1320} < 10^{-5}$$

17 $\int_0^{.5} \frac{1}{\sqrt{1+x^4}}\,dx$ $\quad \frac{1}{\sqrt{1+x}} = (1+x)^{-1/2} = 1 + \left(-\frac{1}{2}\right)x + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}x^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}x^3 + \cdots$

$= 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 + \cdots$

so $\frac{1}{\sqrt{1+x^4}} = 1 - \frac{1}{2}x^4 + \frac{3}{8}x^8 - \frac{5}{16}x^{12} + \cdots,$ AND

$$\int_0^{.5} \frac{1}{\sqrt{1+x^4}}\,dx = \int_0^{.5} \left(1 - \frac{1}{2}x^4 + \frac{3}{8}x^8 - \frac{5}{16}x^{12} + \cdots\right) dx = \left[x - \frac{1}{10}x^5 + \frac{1}{24}x^9 - \frac{5}{208}x^{13} + \cdots\right]_0^{.5}$$

$$= (.5) - \frac{1}{10}(.5)^5 + \frac{1}{24}(.5)^9 - \frac{5}{208}(.5)^{13} + \cdots$$

$$\approx \boxed{.5 - \frac{1}{10}(.5)^5 + \frac{1}{24}(.5)^9} \text{ WITH } |E| < \frac{5}{208}(.5)^{13} < 10^{-5}$$

19 $\int_0^{.1} \frac{\sin x}{x}\,dx$ $\quad \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$ so

$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots$ AND

$$\int_0^{.1} \frac{\sin x}{x}\,dx = \int_0^{.1} \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right) dx = \left[x - \frac{x^3}{18} + \frac{x^5}{600} - \frac{x^7}{7(7!)} + \cdots\right]_0^{.1}$$

$$= .1 - \frac{(.1)^3}{18} + \frac{(.1)^5}{600} - \frac{(.1)^7}{7(5040)} + \cdots \approx \boxed{.1 - \frac{(.1)^3}{18} + \frac{(.1)^5}{600}} \text{ WITH } |E| < \frac{(.1)^7}{35,280} < 10^{-8}$$

20 $\int_0^{.1} e^{-x^2}\,dx$ $\quad e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$ so $e^{-x^2} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$ AND

$$\int_0^{.1} e^{-x^2}\,dx = \int_0^{.1} \left(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \cdots\right) dx = \left[x - \frac{x^3}{3} + \frac{x^5}{10} - \frac{x^7}{42} + \cdots\right]_0^{.1}$$

$$= .1 - \frac{(.1)^3}{3} + \frac{(.1)^5}{10} - \frac{(.1)^7}{42} + \cdots$$

$$\approx \boxed{.1 - \frac{(.1)^3}{3} + \frac{(.1)^5}{10}} \text{ WITH } |E| < \frac{(.1)^7}{42} < 10^{-8}$$