93 $a_1 = -4$, $a_{n+1} = \sqrt{8 + 2a_n}$

ASSUMING THAT $\{a_n\}$ CONVERGES, LET $\lim_{n\to\infty} a_n = L$.

THEN $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{8+2a_n}$ GIVES $\underline{L = \sqrt{8+2L}}$,

SO $L^2 = 8 + 2L$, $L^2 - 2L - 8 = 0$, $(L-4)(L+2) = 0$, $\underline{L=4}$ OR $\underline{L=-2}$.

SINCE $a_n \ge 0$ FOR $n \ge 2$, $L \ge 0$; SO $\boxed{L = 4}$

97 $2,\ 2+\frac{1}{2},\ 2+\cfrac{1}{2+\frac{1}{2}},\ 2+\cfrac{1}{2+\cfrac{1}{2+\frac{1}{2}}},\ \ldots$

WE CAN DEFINE $\{a_n\}$ RECURSIVELY BY $a_1 = 2$ AND $\underline{a_{n+1} = 2 + \frac{1}{a_n}}$ FOR $n \ge 1$.

ASSUMING THAT $\{a_n\}$ CONVERGES, LET $\lim_{n\to\infty} a_n = L$.

THEN $\lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty}\left(2 + \frac{1}{a_n}\right)$ GIVES $\underline{L = 2 + \frac{1}{L}}$,

SO $L^2 = 2L + 1 \Rightarrow L^2 - 2L + \underline{1} = 1 + \underline{1} \Rightarrow (L-1)^2 = 2 \Rightarrow L - 1 = \pm\sqrt{2} \Rightarrow \underline{L = 1 \pm \sqrt{2}}$.

SINCE $a_n > 0$ FOR ALL n, $L \ge 0$; SO $\boxed{L = 1 + \sqrt{2}}$

130 SINCE $\{a_n\}$ CONVERGES, LET $\lim_{n\to\infty} a_n = L$.

IF $\varepsilon > 0$ IS GIVEN, THEN THERE IS A POSITIVE INTEGER N SUCH THAT

IF $n > N$, THEN $|a_n - L| < \frac{\varepsilon}{2}$. (USING $\frac{\varepsilon}{2}$ IN PLACE OF ε)

THEREFORE IF $m, n > N$, THEN

$$|a_m - a_n| = |(a_m - L) + (L - a_n)| \le |a_m - L| + |L - a_n| = |a_m - L| + |a_n - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$

(USING THE <u>TRIANGLE INEQUALITY</u>; SEE PAGE AP-5).

<u>REMARK</u> A SEQUENCE WITH THIS PROPERTY IS A <u>CAUCHY SEQUENCE</u>,

SO THIS PROBLEM SHOWS THAT EVERY CONVERGENT SEQUENCE IS A CAUCHY SEQUENCE.

134 IF $\{a_n\}$ IS A SEQUENCE, $a_n \to 0$ IFF $|a_n| \to 0$.

<u>PF</u> $\Rightarrow$ IF $\lim_{n\to\infty} a_n = 0$, THEN $\lim_{n\to\infty} |a_n| = |0| = 0$ SINCE $f(x) = |x|$ IS CONTINUOUS AT 0.

$\Leftarrow$ IF $\lim_{n\to\infty} |a_n| = 0$, THEN $\lim_{n\to\infty} a_n = 0$ BY THE SQUEEZE TH. SINCE

$-|a_n| \le a_n \le |a_n|$ FOR ALL n AND $\lim_{n\to\infty}(-|a_n|) = 0 = \lim_{n\to\infty} |a_n|$.

OR <u>PF</u> $\lim_{n\to\infty} a_n = 0$ <u>IFF</u> FOR EVERY $\varepsilon > 0$ THERE IS A POSITIVE INTEGER N SUCH THAT IF $n > N$, THEN $|a_n - 0| < \varepsilon$

<u>IFF</u> FOR EVERY $\varepsilon > 0$ THERE IS A POSITIVE INTEGER N SUCH THAT IF $n > N$, THEN $||a_n| - 0| < \varepsilon$

<u>IFF</u> $\lim_{n\to\infty} |a_n| = 0$,

SINCE $||a_n| - 0| = ||a_n|| = |a_n| = |a_n - 0|$.

MATH 21C SOLUTIONS – ORANGE SHEET (SEC. 10.1)

① SHOW THAT $\lim_{n\to\infty} \frac{2}{\sqrt{n}} = 0$.

(PREPARATION: $\left|\frac{2}{\sqrt{n}} - 0\right| < \varepsilon$ IFF $\left|\frac{2}{\sqrt{n}}\right| < \varepsilon$ IFF $\frac{2}{\sqrt{n}} < \varepsilon$ IFF $\frac{4}{n} < \varepsilon^2$ IFF $\frac{n}{4} > \frac{1}{\varepsilon^2}$ IFF $n > \frac{4}{\varepsilon^2}$)

PF LET $\varepsilon > 0$ BE GIVEN, AND LET N BE AN INTEGER WITH $N \geq \frac{4}{\varepsilon^2}$.
IF $n > N$, THEN $n > \frac{4}{\varepsilon^2} \Rightarrow \sqrt{n} > \frac{2}{\varepsilon} \Rightarrow \frac{1}{\sqrt{n}} < \frac{\varepsilon}{2} \Rightarrow \frac{2}{\sqrt{n}} < \varepsilon \Rightarrow$

$$\left|\frac{2}{\sqrt{n}} - 0\right| = \left|\frac{2}{\sqrt{n}}\right| = \frac{2}{\sqrt{n}} < \varepsilon.$$

② SHOW THAT $\lim_{n\to\infty} \frac{6n-8}{2n-1} = 3$.

(PREPARATION: $\left|\frac{6n-8}{2n-1} - 3\right| = \left|\frac{6n-8}{2n-1} - \frac{3(2n-1)}{2n-1}\right| = \left|\frac{6n-8-(6n-3)}{2n-1}\right| = \left|\frac{-5}{2n-1}\right| = \frac{5}{2n-1} < \varepsilon$ IFF $\frac{2n-1}{5} > \frac{1}{\varepsilon}$ IFF $2n-1 > \frac{5}{\varepsilon}$ IFF $2n > 1 + \frac{5}{\varepsilon}$ IFF $n > \frac{1}{2}\left(1 + \frac{5}{\varepsilon}\right)$)

PF LET $\varepsilon > 0$ BE GIVEN, AND LET N BE AN INTEGER WITH $N \geq \frac{1}{2}\left(1 + \frac{5}{\varepsilon}\right)$.
IF $n > N$, THEN $n > \frac{1}{2}\left(1+\frac{5}{\varepsilon}\right) \Rightarrow 2n > 1 + \frac{5}{\varepsilon} \Rightarrow 2n-1 > \frac{5}{\varepsilon} \Rightarrow$

$$\frac{1}{2n-1} < \frac{\varepsilon}{5} \Rightarrow \frac{5}{2n-1} < \varepsilon \Rightarrow \left|\frac{6n-8}{2n-1} - 3\right| = \left|\frac{-5}{2n-1}\right| = \frac{5}{2n-1} < \varepsilon.$$

REMARK WE COULD INSTEAD HAVE CHOSEN N TO BE AN INTEGER WITH $N \geq \frac{5}{\varepsilon}$, USING THE FACT THAT $\frac{5}{2n-1} \leq \frac{5}{2n-n} = \frac{5}{n}$.

③ SHOW THAT $\lim_{n\to\infty} \frac{n^2-2n}{n^2+4} = 1$

(PREPARATION: $\left|\frac{n^2-2n}{n^2+4} - 1\right| = \left|\frac{n^2-2n}{n^2+4} - \frac{n^2+4}{n^2+4}\right| = \left|\frac{-2n-4}{n^2+4}\right| = \frac{2n+4}{n^2+4} \leq \frac{2n+4n}{n^2} = \frac{6n}{n^2} = \frac{6}{n}$, AND $\frac{6}{n} < \varepsilon$ IFF $\frac{n}{6} > \frac{1}{\varepsilon}$ IFF $n > \frac{6}{\varepsilon}$)

PF LET $\varepsilon > 0$ BE GIVEN, AND LET N BE AN INTEGER WITH $N \geq \frac{6}{\varepsilon}$.
IF $n > N$, THEN $n > \frac{6}{\varepsilon} \Rightarrow \frac{n}{6} > \frac{1}{\varepsilon} \Rightarrow \frac{6}{n} < \varepsilon \Rightarrow$

$$\frac{2n+4}{n^2+4} \leq \frac{2n+4n}{n^2} = \frac{6n}{n^2} = \frac{6}{n} < \varepsilon \Rightarrow$$

$$\left|\frac{n^2-2n}{n^2+4} - 1\right| = \left|\frac{-2n-4}{n^2+4}\right| = \frac{2n+4}{n^2+4} < \varepsilon.$$

④ SHOW THAT $\lim_{n\to\infty} r^n = 0$ IF $|r| < 1$.

(PREPARATION: $|r^n - 0| = |r^n| = |r|^n < \varepsilon$ IFF $\ln|r|^n < \ln\varepsilon$ IFF $n\ln|r| < \ln\varepsilon$ IFF $n > \frac{\ln\varepsilon}{\ln|r|}$, SINCE $\ln|r| < 0$)

PF LET $\varepsilon > 0$ BE GIVEN, AND LET N BE AN INTEGER WITH $N \geq \frac{\ln\varepsilon}{\ln|r|}$.
IF $n > N$, THEN $n > \frac{\ln\varepsilon}{\ln|r|} \Rightarrow n\ln|r| < \ln\varepsilon$ (SINCE $\ln|r| < 0$),
SO $\ln|r|^n < \ln\varepsilon \Rightarrow |r|^n < \varepsilon \Rightarrow |r^n| < \varepsilon \Rightarrow |r^n - 0| < \varepsilon$.

REMARK THE STATEMENT IS CLEARLY TRUE FOR $r = 0$, SO THE ABOVE ARGUMENT ASSUMES THAT $r \neq 0$.

(5) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{(n+1)(n+2)} = \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \frac{1}{4\cdot 5} + \cdots + \frac{1}{(n+1)(n+2)} + \cdots$

LET $a_n = \frac{1}{(n+1)(n+2)} = \frac{A}{n+1} + \frac{B}{n+2}$ THEN $1 = A(n+2) + B(n+1)$, SO

$n=-1$: $1 = A$

$n=-2$: $1 = -B$ SO $B = -1$

THEN $S_n = a_1 + \cdots + a_n = \left(\frac{1}{2} - \cancel{\frac{1}{3}}\right) + \left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{4}}\right) + \left(\cancel{\frac{1}{4}} - \cancel{\frac{1}{5}}\right) + \cdots + \left(\cancel{\frac{1}{n+1}} - \frac{1}{n+2}\right) = \frac{1}{2} - \frac{1}{n+2}$ FOR ALL n.

SINCE $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \left(\frac{1}{2} - \frac{1}{n+2}\right) = \frac{1}{2} - 0 = \frac{1}{2}$, THE SERIES $\boxed{\text{CONVERGES}}$ AND ITS SUM $\boxed{S = \frac{1}{2}}$.

(11) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right)$ SINCE $\displaystyle\sum_{n=0}^{\infty} \frac{5}{2^n} = \sum_{n=0}^{\infty} 5\left(\frac{1}{2}\right)^n$ AND $\displaystyle\sum_{n=0}^{\infty} \frac{1}{3^n} = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n$ ARE BOTH CONVERGENT GEOMETRIC SERIES (BECAUSE $|r|<1$), THE SERIES $\boxed{\text{CONVERGES}}$ AND

$$\sum_{n=0}^{\infty} \left(\frac{5}{2^n} + \frac{1}{3^n}\right) = \sum_{n=0}^{\infty} 5\left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{5}{1-\frac{1}{2}} + \frac{1}{1-\frac{1}{3}} = 10 + \frac{3}{2} = \boxed{\frac{23}{2}}$$

(13) $\displaystyle\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right)$ SINCE $\displaystyle\sum_{n=0}^{\infty} \frac{1}{2^n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n$ AND $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{5^n} = \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n$ ARE BOTH CONVERGENT GEOMETRIC SERIES (BECAUSE $|r|<1$), THE SERIES $\boxed{\text{CONVERGES}}$ AND

$$\sum_{n=0}^{\infty} \left(\frac{1}{2^n} + \frac{(-1)^n}{5^n}\right) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n + \sum_{n=0}^{\infty} \left(-\frac{1}{5}\right)^n = \frac{1}{1-1/2} + \frac{1}{1-(-1/5)} = 2 + \frac{5}{6} = \boxed{\frac{17}{6}}$$

(25) $1.24\overline{123} = 1.24123123123\ldots = 1.24 + .00123123123\ldots$

$$= \frac{124}{100} + \frac{1}{100}(.123123123\ldots) = \frac{1}{100}\left[124 + \sum_{n=1}^{\infty} \frac{123}{1000}\left(\frac{1}{1000}\right)^{n-1}\right]$$

GEOM. SERIES WITH $r = .001$, AND $a = .123$

$$= \frac{1}{100}\left[124 + \frac{.123}{1-.001}\right] = \frac{1}{100}\left[124 + \frac{.123}{.999}\right] = \frac{1}{100}\left[124 + \frac{123}{999}\right] = \frac{1}{100}\left[124 + \frac{41}{333}\right] = \boxed{\frac{41,333}{33,300}}$$

(41) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{(4n-3)(4n+1)} = \frac{4}{1\cdot 5} + \frac{4}{5\cdot 9} + \frac{4}{9\cdot 13} + \cdots + \frac{4}{(4n-3)(4n+1)} + \cdots$

LET $a_n = \frac{4}{(4n-3)(4n+1)} = \frac{A}{4n-3} + \frac{B}{4n+1}$, SO $4 = A(4n+1) + B(4n-3)$

$n = 3/4$: $4 = 4A$ $A = 1$

$n = -1/4$: $4 = -4B$ $B = -1$

THEN $S_n = a_1 + \cdots + a_n = \left(\frac{1}{1} - \cancel{\frac{1}{5}}\right) + \left(\cancel{\frac{1}{5}} - \cancel{\frac{1}{9}}\right) + \left(\cancel{\frac{1}{9}} - \cancel{\frac{1}{13}}\right) + \cdots + \left(\cancel{\frac{1}{4n-3}} - \frac{1}{4n+1}\right) = 1 - \frac{1}{4n+1}$ FOR ALL n.

SINCE $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \left(1 - \frac{1}{4n+1}\right) = 1 - 0 = 1$,

THE SERIES CONVERGES WITH SUM $\boxed{S = 1}$.

Sec. 10.2 – SOLUTIONS

(51) $\sum_{n=1}^{\infty} (-1)^{n+1}\frac{3}{2^n} = \sum_{n=1}^{\infty}(-3)\left(-\frac{1}{2}\right)^n$ IS A GEOMETRIC SERIES WITH $|r|<1$, SO IT CONVERGES WITH SUM $S = \frac{a}{1-r} = \frac{3/2}{1-(-1/2)} = \boxed{1}$.

(59) $\sum_{n=0}^{\infty} \frac{2^n-1}{3^n} = \sum_{n=0}^{\infty}\left(\left(\frac{2}{3}\right)^n - \left(\frac{1}{3}\right)^n\right)$ CONVERGES SINCE $\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n$ AND $\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n$ ARE CONVERGENT GEOM. SERIES (BECAUSE $|r|<1$), AND

$$\sum_{n=0}^{\infty} \frac{2^n-1}{3^n} = \sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n - \sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{1-2/3} - \frac{1}{1-1/3} = 3 - \frac{3}{2} = \boxed{\frac{3}{2}}$$

(60) $\sum_{n=1}^{\infty}\left(1-\frac{1}{n}\right)^n$ $\lim_{n\to\infty}\left(1-\frac{1}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{-1}{n}\right)^n = e^{-1} \neq 0$,

SO THE SERIES $\boxed{\text{DIVERGES}}$ BY THE DIVERGENCE TEST.

(63) $\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty}\left(\frac{2^n}{4^n}+\frac{3^n}{4^n}\right) = \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right)$ CONVERGES SINCE $\sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n$ AND $\sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n$ ARE CONVERGENT GEOM. SERIES (BECAUSE $|r|<1$), AND

$$\sum_{n=1}^{\infty}\frac{2^n+3^n}{4^n} = \sum_{n=1}^{\infty}\left(\left(\frac{1}{2}\right)^n + \left(\frac{3}{4}\right)^n\right) = \sum_{n=1}^{\infty}\left(\frac{1}{2}\right)^n + \sum_{n=1}^{\infty}\left(\frac{3}{4}\right)^n = \frac{1/2}{1-1/2} + \frac{3/4}{1-3/4} = 1+3 = \boxed{4}$$

(64) $\sum_{n=1}^{\infty}\frac{2^n+4^n}{3^n+4^n}$ $\lim_{n\to\infty}\frac{2^n+4^n}{3^n+4^n} \div \frac{4^n}{4^n} = \lim_{n\to\infty}\frac{\frac{2^n}{4^n}+1}{\frac{3^n}{4^n}+1} = \lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)^n+1}{\left(\frac{3}{4}\right)^n+1} = \frac{0+1}{0+1} = 1 \neq 0$,

SO THE SERIES $\boxed{\text{DIVERGES}}$ BY THE DIVERGENCE TEST.

(65) $\sum_{n=1}^{\infty}\ln\left(\frac{n}{n+1}\right) = \sum_{n=1}^{\infty}\left[\ln n - \ln(n+1)\right]$

$$\underline{S_n} = a_1 + \dots + a_n = [\ln 1 - \cancel{\ln 2}] + [\cancel{\ln 2} - \cancel{\ln 3}] + [\cancel{\ln 3} - \cancel{\ln 4}] + \dots + [\cancel{\ln n} - \ln(n+1)]$$
$$= \ln 1 - \ln(n+1) = 0 - \ln(n+1) = \underline{-\ln(n+1)},$$

SO $\underline{\lim_{n\to\infty} S_n} = \lim_{n\to\infty}(-\ln(n+1)) = \underline{-\infty}$ AND THEREFORE THE SERIES $\boxed{\text{DIVERGES}}$.

(66) $\sum_{n=1}^{\infty}\ln\left(\frac{n}{2n+1}\right)$ $\lim_{n\to\infty}\ln\left(\frac{n}{2n+1}\right) = \ln\frac{1}{2} \neq 0$ (SINCE $\lim_{n\to\infty}\frac{n}{2n+1} = \frac{1}{2}$, AND $f(x) = \ln x$ IS CONTINUOUS AT $\frac{1}{2}$),

SO THE SERIES $\boxed{\text{DIVERGES}}$ BY THE DIVERGENCE TEST.