MATH 21C ANSWERS – TEST 1

① A) $\sum_{n=1}^{\infty}\left(\frac{5(3^n)}{4^n}-\frac{2}{\sqrt{n}}\right)$

i) $\sum_{n=1}^{\infty}\frac{5(3^n)}{4^n}=\sum_{n=1}^{\infty}5\left(\frac{3}{4}\right)^n$ CONVERGES since it is a geometric series with $|r|<1$, and

ii) $\sum_{n=1}^{\infty}\frac{2}{\sqrt{n}}$ DIVERGES since it is a multiple of a p-series with $p\le 1$.

Therefore $\sum_{n=1}^{\infty}\left(\frac{5(3^n)}{4^n}-\frac{2}{\sqrt{n}}\right)$ $\boxed{\text{DIVERGES}}$ (since one series converges and the other one diverges)

B) $\sum_{n=1}^{\infty}\frac{4^n n^8}{5^{n+3}}$ $\quad \ell=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{4^{n+1}(n+1)^8}{5^{n+4}}\cdot\frac{5^{n+3}}{4^n n^8}=\lim_{n\to\infty}\frac{4}{5}\left(\frac{n+1}{n}\right)^8=\frac{4}{5}\cdot 1^8=\frac{4}{5}<1,$

so the series $\boxed{\text{CONVERGES}}$ by the Ratio Test.

$\boxed{\text{OR}}$ $\ell=\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\left(\frac{4^n n^8}{5^n\cdot 5^3}\right)^{1/n}=\lim_{n\to\infty}\frac{4\cdot n^{8/n}}{5\cdot 5^{3/n}}=\lim_{n\to\infty}\frac{4}{5}\cdot\frac{(n^{1/n})^8}{5^{3/n}}$

$=\frac{4}{5}\cdot\frac{1^8}{5^0}=\frac{4}{5}<1,$ so the series $\boxed{\text{CONVERGES}}$ by the Root Test.

C) $\sum_{n=1}^{\infty}\frac{n^3+9}{n^4+5n}$ compare to $\sum_{n=1}^{\infty}\frac{1}{n}$, which diverges since it's the harmonic series;

$\frac{n^3+9}{n^4+5n}\ge\frac{1}{n}$ iff $n^4+9n\ge n^4+5n$ iff $9n\ge 5n$, and this is true for all $n\ge 1$;

so the series $\boxed{\text{DIVERGES}}$ by the Comparison Test.

$\boxed{\text{OR}}$ use $\lim_{n\to\infty}\frac{n^3+9}{n^4+5n}\div\frac{1}{n}=\lim_{n\to\infty}\frac{n^4+9n}{n^4+5n}\ne 0,$ so the series $\boxed{\text{DIVERGES}}$ by the LCT.

D) $\sum_{n=1}^{\infty}\left(\frac{n+2}{n+8}\right)^n$ $\quad \lim_{n\to\infty}\left(\frac{n+2}{n+8}\right)^n=\lim_{n\to\infty}\left(\frac{1+\frac{2}{n}}{1+\frac{8}{n}}\right)^n=\lim_{n\to\infty}\frac{(1+\frac{2}{n})^n}{(1+\frac{8}{n})^n}=\frac{e^2}{e^8}=e^{-6}\ne 0,$

so the series $\boxed{\text{DIVERGES}}$ by the Divergence Test.

E) $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ $\quad \ln n<\sqrt{n}$ for all n,

so $\frac{\ln n}{n^2}<\frac{\sqrt{n}}{n^2}=\frac{1}{n^{3/2}}$ for all n;

and since $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ converges since it's a p-series with $p>1$,

$\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ $\boxed{\text{CONVERGES}}$ by the Comparison Test.

$\boxed{\text{OR}}$ use $\lim_{n\to\infty}\frac{\ln n}{n^2}\div\frac{1}{n^{3/2}}=\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}=\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}\underset{(\frac{\infty}{\infty})}{=}\lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0\ne\infty,$

so the series $\boxed{\text{CONVERGES}}$ by the LCT.

F) $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ $\quad f(x)=\frac{1}{x(\ln x)^2}$ is cont. and decreasing for $x>1$, and

$(u=\ln x,\ du=\frac{1}{x}dx)$

$\int_2^{\infty}\frac{1}{x(\ln x)^2}dx=\lim_{T\to\infty}\int_2^T\frac{1}{x(\ln x)^2}dx=\lim_{T\to\infty}\int_{x=2}^{x=T}\frac{1}{u^2}du$

$=\lim_{T\to\infty}\left[-\frac{1}{u}\right]_{x=2}^{x=T}=\lim_{T\to\infty}\left[-\frac{1}{\ln x}\right]_2^T=\lim_{T\to\infty}\left[-\frac{1}{\ln T}-\left(-\frac{1}{\ln 2}\right)\right]=\frac{1}{\ln 2};$

$\left(\frac{1}{\ln T}\to 0 \text{ as } T\to\infty\right)$

so $\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^2}$ $\boxed{\text{CONVERGES}}$ by the Integral Test.

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(2) $$\sum_{n=1}^{\infty}\left[\frac{14(3^{n-1})}{5^n}+(-1)^{n+1}\frac{30}{2^{n+1}}\right]=\sum_{n=1}^{\infty}\frac{14\cdot 3^n\cdot 3^{-1}}{5^n}+\sum_{n=1}^{\infty}30\left(-\frac{1}{2}\right)^{n+1}$$

$$=\sum_{n=1}^{\infty}\frac{14}{3}\left(\frac{3}{5}\right)^n+\sum_{n=1}^{\infty}30\left(-\frac{1}{2}\right)^{n+1}=\frac{14/5}{1-3/5}+\frac{15/2}{1-(-1/2)}=\frac{14/5}{2/5}+\frac{15/2}{3/2}=7+5=\boxed{12}$$

(3) A) $\lim_{n\to\infty} S_n=\lim_{n\to\infty}\frac{3n}{n+5}=3$, so the series $\boxed{\text{CONVERGES}}$ (AND HAS SUM $S=3$).

B) $$a_n=S_n-S_{n-1}=\frac{3n}{n+5}-\frac{3(n-1)}{(n-1)+5}=\frac{3n}{n+5}-\frac{3n-3}{n+4}=\frac{3n^2+12n-(3n^2+12n-15)}{(n+5)(n+4)}=\boxed{\frac{15}{(n+5)(n+4)}}$$

(4) A) $$\sum_{n=1}^{\infty}(-1)^n\frac{n+2}{n^2+32}$$

a) $\sum_{n=1}^{\infty}|a_n|=\sum_{n=1}^{\infty}\frac{n+2}{n^2+32}$ COMPARE TO $\sum_{n=1}^{\infty}\frac{1}{n}$, WHICH DIVERGES (HARMONIC SERIES)!

$\frac{n+2}{n^2+32}\geq\frac{1}{n}$ IFF $n^2+2n\geq n^2+32$ IFF $2n\geq 32$ IFF $n\geq 16$,

so $\sum_{n=1}^{\infty}|a_n|$ DIVERGES BY THE COMPARISON TEST.

OR USE $\lim_{n\to\infty}\frac{n+2}{n^2+32}\div\frac{1}{n}=\lim_{n\to\infty}\frac{n^2+2n}{n^2+32}=1\neq 0$, so $\sum_{n=1}^{\infty}|a_n|$ DIVERGES BY THE LCT.

b) 1) $\lim_{n\to\infty}\frac{n+2}{n^2+32}=0$ (SINCE THE DEGREE ON TOP IS LESS THAN THE DEGREE ON THE BOTTOM)

2) $\frac{n+2}{n^2+32}\geq\frac{n+3}{(n+1)^2+32}$ IFF $\frac{n+2}{n^2+32}\geq\frac{n+3}{n^2+2n+33}$ IFF $(n+2)(n^2+2n+33)\geq(n+3)(n^2+32)$

IFF $n^3+4n^2+37n+66\geq n^3+3n^2+32n+96$ IFF $n^2+5n\geq 30$ IFF $n(n+5)\geq 30$ IFF $n\geq 4$.

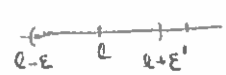
THEREFORE THE SERIES CONVERGES BY THE AST,
so IT IS $\boxed{\text{CONDITIONALLY CONVERGENT}}$.

OR 2) $\frac{n+2}{n^2+32}\geq\frac{n+3}{(n+1)^2+32}$ FOR $n\geq 4$ SINCE IF $f(x)=\frac{x+2}{x^2+32}$,

$$f'(x)=\frac{(x^2+32)-(x+2)(2x)}{(x^2+32)^2}=\frac{-x^2-4x+32}{(x^2+32)^2}=\frac{(x+8)(4-x)}{(x^2+32)^2}<0 \text{ IF } x>4;$$

so f IS DECREASING FOR $x\geq 4$.

(5) LET $\lim_{n\to\infty}(a_n)^{1/n}=\ell$ WHERE $\ell<1$, AND LET $\varepsilon>0$ WITH $\varepsilon<1-\ell$.



THEN $\ell-\varepsilon<(a_n)^{1/n}<\ell+\varepsilon<1$ FOR $n\geq N$ (FOR SOME N),

so $a_n<(\ell+\varepsilon)^n$ FOR $n\geq N$.

SINCE $\sum_{n=1}^{\infty}(\ell+\varepsilon)^n$ IS A CONVERGENT GEOMETRIC SERIES (BECAUSE $|r|=\ell+\varepsilon<1$),

$\sum_{n=1}^{\infty}a_n$ CONVERGES BY THE COMPARISON TEST.

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(4) B) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{4^n n!}{(n+2)^n}$

a) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{4^n n!}{(n+2)^n}$

$$\ell = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{4^{n+1}(n+1)!}{(n+3)^{n+1}} \cdot \frac{(n+2)^n}{\cancel{4^n n!}} = \lim_{n\to\infty} \frac{4(n+1)(n+2)^n}{(n+3)^{n+1}}$$

$$= \lim_{n\to\infty} 4 \cdot \frac{n+1}{n+3} \cdot \frac{(n+2)^n}{(n+3)^n} = \lim_{n\to\infty} 4 \cdot \frac{n+1}{n+3} \cdot \left(\frac{n+2}{n+3}\right)^n$$

$$= \lim_{n\to\infty} 4 \cdot \frac{n+1}{n+3} \cdot \frac{\left(1+\frac{2}{n}\right)^n}{\left(1+\frac{3}{n}\right)^n} = 4 \cdot 1 \cdot \frac{e^2}{e^3} = \frac{4}{e} > 1,$$

so $\sum_{n=1}^{\infty} |a_n|$ diverges by the Ratio Test and $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{DIVERGES}}$ also

(since $\lim_{n\to\infty} a_n \neq 0$).

(4) c) $\sum_{n=1}^{\infty} (-1)^n \frac{5^n + 4^{n+1}}{8^n - 5^n}$

a) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{5^n + 4^{n+1}}{8^n - 5^n}$ compare to $\sum_{n=1}^{\infty} \frac{5^n}{8^n} = \sum_{n=1}^{\infty} \left(\frac{5}{8}\right)^n$, which converges since it's a geometric series with $|r| < 1$:

$$\lim_{n\to\infty} \frac{5^n + 4^{n+1}}{8^n - 5^n} \div \frac{5^n}{8^n} = \lim_{n\to\infty} \frac{1 + \frac{4^{n+1}}{5^n}}{1 - \frac{5^n}{8^n}} = \lim_{n\to\infty} \frac{1 + 4\left(\frac{4}{5}\right)^n}{1 - \left(\frac{5}{8}\right)^n} = \frac{1+0}{1-0} = 1 \neq \infty,$$

so $\sum_{n=1}^{\infty} |a_n|$ converges by the LCT and $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

OR use $\ell = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{5^{n+1} + 4^{n+2}}{8^{n+1} - 5^{n+1}} \cdot \frac{8^n - 5^n}{5^n + 4^{n+1}}$

$$= \lim_{n\to\infty} \frac{5^{n+1} + 4^{n+2}}{5^n + 4^{n+1}} \cdot \frac{8^n - 5^n}{8^{n+1} - 5^{n+1}} = \lim_{n\to\infty} \frac{5 + 16\left(\frac{4}{5}\right)^n}{1 + 4\left(\frac{4}{5}\right)^n} \cdot \frac{1 - \left(\frac{5}{8}\right)^n}{8 - 5\left(\frac{5}{8}\right)^n} = 5 \cdot \frac{1}{8} = \frac{5}{8} < 1,$$

so $\sum_{n=1}^{\infty} |a_n|$ converges by the Ratio Test and $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

OR use $\ell = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(\frac{5^n + 4^{n+1}}{8^n - 5^n}\right)^{1/n} = \lim_{n\to\infty} \frac{\left(5^n\left(1 + \frac{4^{n+1}}{5^n}\right)\right)^{1/n}}{\left(8^n\left(1 - \frac{5^n}{8^n}\right)\right)^{1/n}}$

$$= \lim_{n\to\infty} \frac{5\left(1 + 4\left(\frac{4}{5}\right)^n\right)^{1/n}}{8\left(1 - \left(\frac{5}{8}\right)^n\right)^{1/n}} = \frac{5 \cdot 1^0}{8 \cdot 1^0} = \frac{5}{8} < 1,$$

so $\sum_{n=1}^{\infty} |a_n|$ converges by the Root Test and $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.
