① A) $\sum_{n=1}^{\infty}\left(\frac{1}{4n}-\frac{9}{5^n}\right)$

1) $\sum_{n=1}^{\infty}\frac{1}{4n}$ DIVERGES (MULTIPLE OF THE HARMONIC SERIES) AND

2) $\sum_{n=1}^{\infty}\frac{9}{5^n}=\sum_{n=1}^{\infty}9\left(\frac{1}{5}\right)^n$ CONVERGES (GEOMETRIC SERIES WITH $|r|<1$),

SO THE SERIES **DIVERGES** (SINCE ONE SERIES DIVERGES AND THE OTHER CONVERGES).

B) $\sum_{n=1}^{\infty}\frac{n^2+8}{n^4+2n^3}$ COMPARE TO $\sum_{n=1}^{\infty}\frac{1}{n^2}$, WHICH CONVERGES (P-SERIES, $P>1$):

$\frac{n^2+8}{n^4+2n^3}\le\frac{1}{n^2}$ IFF $n^4+8n^2\le n^4+2n^3$ IFF $8n^2\le 2n^3$ IFF $n\ge 4$,

SO THE SERIES **CONVERGES** BY THE COMPARISON TEST.

(OR USE $\lim_{n\to\infty}\frac{n^2+8}{n^4+2n^3}\div\frac{1}{n^2}=\lim_{n\to\infty}\frac{n^4+8n^2}{n^4+2n^3}=1\ne\infty$,

SO THE SERIES **CONVERGES** BY THE LIMIT COMPARISON TEST)

C) $\sum_{n=1}^{\infty}\frac{(n+5)^9}{2^n}$ $\rho=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{(n+6)^9}{2^{n+1}}\cdot\frac{2^n}{(n+5)^9}=\lim_{n\to\infty}\frac{1}{2}\left(\frac{n+6}{n+5}\right)^9=\frac{1}{2}\cdot 1^9=\frac{1}{2}<1$,

SO THE SERIES **CONVERGES** BY THE RATIO TEST.

D) $\sum_{n=1}^{\infty}\frac{\sqrt{n}+2}{n+5\sqrt{n}}$ COMPARE TO $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, WHICH DIVERGES (P-SERIES, $P\le 1$):

$\lim_{n\to\infty}\frac{\sqrt{n}+2}{n+5\sqrt{n}}\div\frac{1}{\sqrt{n}}=\lim_{n\to\infty}\frac{n+2\sqrt{n}}{n+5\sqrt{n}}=\lim_{n\to\infty}\frac{1+\frac{2}{\sqrt{n}}}{1+\frac{5}{\sqrt{n}}}=1\ne 0$,

SO THE SERIES **DIVERGES** BY THE LIMIT COMPARISON TEST.

E) $\sum_{n=1}^{\infty}\left(\frac{2n+1}{2n+5}\right)^n$ $\lim_{n\to\infty}\left(\frac{2n+1}{2n+5}\right)^n=\lim_{n\to\infty}\left(\frac{1+\frac{1}{2n}}{1+\frac{5}{2n}}\right)^n=\lim_{n\to\infty}\frac{\left(1+\frac{1/2}{n}\right)^n}{\left(1+\frac{5/2}{n}\right)^n}=\frac{e^{1/2}}{e^{5/2}}=e^{-2}\ne 0$,

SO THE SERIES **DIVERGES** BY THE DIVERGENCE TEST.

F) $\sum_{n=1}^{\infty}\left(\frac{3n+8}{4n-3}\right)^n$ $\rho=\lim_{n\to\infty}(a_n)^{1/n}=\lim_{n\to\infty}\frac{3n+8}{4n-3}=\frac{3}{4}<1$,

SO THE SERIES **CONVERGES** BY THE ROOT TEST.

② $\lim_{n\to\infty}\frac{(\ln n)^2}{n^3}=\lim_{x\to\infty}\frac{(\ln x)^2}{x^3}\ \left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{2\ln x\cdot\frac{1}{x}}{3x^2}=\lim_{x\to\infty}\frac{2\ln x}{3x^3}\ \left(\frac{\infty}{\infty}\right)=\lim_{x\to\infty}\frac{2\cdot\frac{1}{x}}{9x^2}$

$=\lim_{x\to\infty}\frac{2}{9x^3}=\boxed{0}$

③ $\sum_{n=1}^{\infty}\left(\frac{9}{2^n}+(-1)^{n+1}\frac{20}{3^{n-1}}\right)=\sum_{n=1}^{\infty}9\left(\frac{1}{2}\right)^n+\sum_{n=1}^{\infty}(-60)\left(-\frac{1}{3}\right)^n$

$=\frac{9/2}{1-1/2}+\frac{20}{1-(-1/3)}=9+\frac{20}{4/3}=9+15=\boxed{24}$

(USING $S=\frac{a}{1-r}$)

(4) A) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n+1}{(n+6)^2}$

1) $\displaystyle\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n+1}{(n+6)^2}$ COMPARE TO $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$, WHICH DIVERGES (HARMONIC SERIES):

$$\lim_{n\to\infty} \frac{n+1}{(n+6)^2} \div \frac{1}{n} = \lim_{n\to\infty} \frac{n^2+n}{n^2+12n+36} = 1 \neq 0,$$

SO $\displaystyle\sum_{n=1}^{\infty} |a_n|$ DIVERGES BY THE LCT.

2) i) $\displaystyle\lim_{n\to\infty} \frac{n+1}{(n+6)^2} = 0$ (SINCE THE DEGREE ON TOP IS LESS THAN THE DEGREE OF THE BOTTOM).

ii) $\dfrac{n+1}{(n+6)^2} \geq \dfrac{n+2}{(n+7)^2}$ IFF $(n+1)(n+7)^2 \geq (n+2)(n+6)^2$

IFF $(n+1)(n^2+14n+49) \geq (n+2)(n^2+12n+36)$

IFF $n^3+15n^2+63n+49 \geq n^3+14n^2+60n+72$ IFF $n^2+3n \geq 23$

IFF $n(n+3) \geq 23$ IFF $n \geq 4$.

THEREFORE THE ORIGINAL SERIES CONVERGES BY THE AST,

SO IT IS **CONDITIONALLY CONVERGENT**.

OR ii) USE THAT IF $f(x) = \dfrac{x+1}{(x+6)^2}$, THEN $f'(x) = \dfrac{4-x}{(x+6)^3} < 0$ FOR $x > 4$;

SO f IS DECREASING FOR $x \geq 4$ AND THEREFORE

$\dfrac{n+1}{(n+6)^2} \geq \dfrac{n+2}{(n+7)^4}$ FOR $n \geq 4$.

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(4) B) $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{4^n n^8}{5^n - 3^n}$

1) $\displaystyle\sum_{n=0}^{\infty} |a_n| = \sum_{n=0}^{\infty} \frac{4^n n^8}{5^n - 3^n}$

$$\ell = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{4^{n+1}(n+1)^8}{5^{n+1}-3^{n+1}} \cdot \frac{5^n - 3^n}{4^n n^8}$$

$$= \lim_{n\to\infty} 4 \cdot \left(\frac{n+1}{n}\right)^8 \cdot \frac{5^n - 3^n}{5^{n+1} - 3^{n+1}} \quad \leftarrow \left(\text{DIVIDE BY } \frac{5^n}{5^n}\right)$$

$$= \lim_{n\to\infty} 4 \cdot \left(\frac{n+1}{n}\right)^8 \cdot \frac{1 - \left(\frac{3}{5}\right)^n}{5 - 3\left(\frac{3}{5}\right)^n} = 4 \cdot 1^8 \cdot \frac{1}{5} = \frac{4}{5} < 1,$$

SO THE SERIES **CONVERGES ABSOLUTELY** BY THE RATIO TEST.

OR $$\ell = \lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{4 n^{8/n}}{(5^n - 3^n)^{1/n}} = \lim_{n\to\infty} \frac{4 (n^{1/n})^8}{\left(5^n \left(1 - \frac{3^n}{5^n}\right)\right)^{1/n}}$$

$$= \lim_{n\to\infty} \frac{4 (n^{1/n})^8}{5\left(1 - \left(\frac{3}{5}\right)^n\right)^{1/n}} = \frac{4 \cdot 1^8}{5 \cdot 1^0} = \frac{4}{5} < 1,$$

SO THE SERIES **CONVERGES ABSOLUTELY** BY THE ROOT TEST.

(4) c) $\sum_{n=1}^{\infty} (-1)^n \frac{n^n}{2^n (n+4)!}$

i) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n^n}{2^n (n+4)!}$

$$\ell = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{2^{n+1}(n+5)!} \cdot \frac{2^n (n+4)!}{n^n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{2(n+5)\, n^n}$$

$$= \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n+5} \cdot \left(\frac{n+1}{n}\right)^n = \lim_{n\to\infty} \frac{1}{2} \cdot \frac{n+1}{n+5} \cdot \left(1+\frac{1}{n}\right)^n = \frac{1}{2}\cdot 1 \cdot e = \frac{e}{2} > 1,$$

so $\sum_{n=1}^{\infty} |a_n|$ DIVERGES BY THE RATIO TEST AND THEREFORE $\sum_{n=1}^{\infty} a_n$ **DIVERGES** ALSO (SINCE $\lim_{n\to\infty} a_n \neq 0$).

(5) $\sum_{n=1}^{\infty} \frac{2n^2+4n+1}{n^2(n+1)^2}$ $\qquad$ $\frac{2n^2+4n+1}{n^2(n+1)^2} = \frac{A}{n} + \frac{B}{n^2} + \frac{C}{n+1} + \frac{D}{(n+1)^2}$

$$2n^2+4n+1 = An(n+1)^2 + B(n+1)^2 + Cn^2(n+1) + Dn^2$$

$n=0$:	$1 = B$
$n=-1$:	$-1 = D$
COEFF. OF n:	$4 = A + 2B = A + 2$ so $A = 2$
COEFF. OF n^3:	$0 = A + C$ so $C = -2$

$$S_n = a_1 + a_2 + a_3 + \cdots + a_n$$

$$= \left(\frac{2}{1} - \frac{2}{2} + \frac{1}{1^2} - \frac{1}{2^2}\right) + \left(\frac{2}{2} - \frac{2}{3} + \frac{1}{2^2} - \frac{1}{3^2}\right) + \cdots + \left(\frac{2}{n} - \frac{2}{n+1} + \frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$= \left(\frac{2}{1} - \frac{2}{2} + \frac{2}{2} - \frac{2}{3} + \cdots + \frac{2}{n} - \frac{2}{n+1}\right) + \left(\frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{2^2} - \frac{1}{3^2} + \cdots + \frac{1}{n^2} - \frac{1}{(n+1)^2}\right)$$

$$= 2 - \frac{2}{n+1} + 1 - \frac{1}{(n+1)^2} = 3 - \frac{2}{n+1} - \frac{1}{(n+1)^2}$$

SINCE $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(3 - \frac{2}{n+1} - \frac{1}{(n+1)^2}\right) = 3$,

THE SERIES CONVERGES WITH SUM $\boxed{S = 3}$.