MATH 21C

ANSWERS – TEST 2 REVIEW PROBLEMS

① $\vec{u} = \langle 4, 8, 1\rangle$ AND $\vec{v} = \langle 2, 1, 2\rangle$

A) $\cos\theta = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|} = \dfrac{18}{\sqrt{81}\,\sqrt{9}} = \dfrac{18}{9\cdot 3} = \boxed{\dfrac{2}{3}}$

B) $\text{proj}_{\vec{v}}\vec{u} = \left(\dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \dfrac{18}{9}\vec{v} = 2\vec{v} = \boxed{\langle 4, 2, 4\rangle}$

② 1) $L = \lim\limits_{n\to\infty}\dfrac{|u_{n+1}|}{|u_n|} = \lim\limits_{n\to\infty}\dfrac{|x+2|^{n+1}}{(n+5)3^{n+1}}\cdot\dfrac{(n+4)3^n}{|x+2|^n} = \lim\limits_{n\to\infty}\dfrac{1}{3}\cdot\dfrac{n+4}{n+5}\cdot|x+2| = \dfrac{1}{3}\cdot 1\cdot|x+2| = \dfrac{|x+2|}{3}$

2) $\underline{L<1}$ IFF $\dfrac{|x+2|}{3} < 1$ IFF $|x+2| < 3$ IFF $-3 < x+2 < 3$ IFF $\underline{-5 < x < 1}$

3) $\underline{x=1}$: $\displaystyle\sum_{n=0}^{\infty}\frac{1}{n+4}$ DIVERGES (HARMONIC SERIES WITH 3 TERMS DELETED)

$\underline{x=-5}$: $\displaystyle\sum_{n=0}^{\infty}(-1)^n\frac{1}{n+4}$ CONVERGES BY THE AST SINCE i) $\lim\limits_{n\to\infty}\dfrac{1}{n+4} = 0$ AND ii) $\dfrac{1}{n+4} \geq \dfrac{1}{n+5}$ FOR ALL n

THEREFORE $\boxed{[-5, 1)}$ IS THE INTERVAL OF CONVERGENCE.

③ $P(1,1,2)$, $Q(2,5,4)$, $R(5,6,5)$

LET $\vec{a} = \vec{PQ} = \langle 1, 4, 2\rangle$ AND $\vec{b} = \vec{PR} = \langle 4, 5, 3\rangle$

THEN $\vec{n} = \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 4 & 2 \\ 4 & 5 & 3 \end{vmatrix} = 2\vec{i} - (-5)\vec{j} - 11\vec{k} = \langle 2, 5, -11\rangle$,

SO THE PLANE HAS EQUATION $2x + 5y - 11z = 2(1) + 5(1) - 11(2)$ (USING P)

OR $\boxed{2x + 5y - 11z = -15}$

④ $\lim\limits_{(x,y)\to(0,0)}\dfrac{8xy}{x^2+y^2}$

1) ON THE X-AXIS, $\lim\limits_{(x,y)\to(0,0)}\dfrac{8xy}{x^2+y^2} = \lim\limits_{x\to 0}\dfrac{0}{x^2} = \lim\limits_{x\to 0} 0 = \underline{0}$

2) ON THE LINE $y = x$, $\lim\limits_{(x,y)\to(0,0)}\dfrac{8xy}{x^2+y^2} = \lim\limits_{x\to 0}\dfrac{8x^2}{2x^2} = \lim\limits_{x\to 0} 4 = \underline{4}$

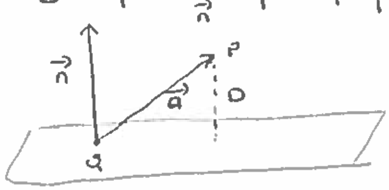
THEREFORE $\lim\limits_{(x,y)\to(0,0)}\dfrac{8xy}{x^2+y^2}$ $\boxed{\text{DOES NOT EXIST}}$.

⑤ DISTANCE FROM $P(8,5,11)$ TO THE PLANE $2x - 6y - 3z = 23$:

$D = \dfrac{|2(8) - 6(5) - 3(11) - 23|}{\sqrt{2^2+6^2+3^2}} = \dfrac{|16 - 30 - 33 - 23|}{\sqrt{49}} = \dfrac{70}{7} = \boxed{10}$

OR LET $Q = (10, 0, -1)$ BE A POINT ON THE PLANE, LET $\vec{a} = \vec{QP} = \langle -2, 5, 12\rangle$, AND LET $\vec{n} = \langle 2, -6, -3\rangle$. THEN

$D = \left|\text{comp}_{\vec{n}}\vec{a}\right| = \left|\dfrac{\vec{a}\cdot\vec{n}}{|\vec{n}|}\right| = \dfrac{|\vec{a}\cdot\vec{n}|}{|\vec{n}|} = \dfrac{|-70|}{\sqrt{49}} = \dfrac{70}{7} = \boxed{10}$



(6) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

$e^{-x^5} = 1 + (-x^5) + \frac{(-x^5)^2}{2!} + \frac{(-x^5)^3}{3!} + \frac{(-x^5)^4}{4!} + \cdots = 1 - x^5 + \frac{x^{10}}{2} - \frac{x^{15}}{6} + \frac{x^{20}}{24} - \cdots$

$x^3 e^{-x^5} = x^3 - x^8 + \frac{x^{13}}{2} - \frac{x^{18}}{6} + \frac{x^{23}}{24} - \cdots$

so $\int_0^{.6} x^3 e^{-x^5}\,dx = \int_0^{.6} \left(x^3 - x^8 + \frac{x^{13}}{2} - \frac{x^{18}}{6} + \frac{x^{23}}{24} - \cdots\right) dx = \left[\frac{x^4}{4} - \frac{x^9}{9} + \frac{x^{14}}{28} - \frac{x^{19}}{114} + \frac{x^{24}}{576} - \cdots\right]_0^{.6}$

$= \frac{(.6)^4}{4} - \frac{(.6)^9}{9} + \frac{(.6)^{14}}{28} - \frac{(.6)^{19}}{114} + \frac{(.6)^{24}}{576} - \cdots$

$\approx \boxed{\frac{(.6)^4}{4} - \frac{(.6)^9}{9} + \frac{(.6)^{14}}{28} - \frac{(.6)^{19}}{114}}$ WITH $|E| < \frac{(.6)^{24}}{576}$

(7) LET $\vec{a} = \vec{PQ} = \langle 3, -4, -2 \rangle$, SO THE LINE HAS PARAMETRIC EQUATIONS

$x = 2 + 3t,\ y = 7 - 4t,\ z = 1 - 2t$ (WITH $t \in \mathbb{R}$)

SUBSTITUTING INTO THE EQUATION OF THE PLANE GIVES

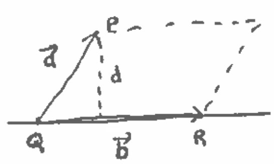
$2(2+3t) + 3(7-4t) - (1-2t) = 44$, SO $4 + 6t + 21 - 12t - 1 + 2t = 44$,

$-4t = 20$, $t = -5$; $\boxed{x = -13,\ y = 27,\ z = 11}$

(8) $\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \cdots$

so $\sqrt{1.04} \approx 1 + \frac{1}{2}(.04) - \frac{1}{8}(.04)^2 + \frac{1}{16}(.04)^3 = \boxed{1 + .02 - .0002 + .000004} = 1.019804$

$= \boxed{1 + \frac{1}{50} - \frac{1}{5{,}000} + \frac{1}{250{,}000}}$

(9) 

LET $\vec{a} = \vec{QP} = \langle 6, 5, 2 \rangle$ AND $\vec{b} = \vec{QR} = \langle 1, 2, -2 \rangle$

so $\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 6 & 5 & 2 \\ 1 & 2 & -2 \end{vmatrix} = \langle -14, 14, 7 \rangle = 7\langle -2, 2, 1 \rangle$

$d = \frac{|\vec{a} \times \vec{b}|}{|\vec{b}|} = \frac{7\sqrt{9}}{\sqrt{9}} = \frac{7 \cdot 3}{3} = \boxed{7}$

$\boxed{\text{OR}}$ $\text{comp}_{\vec{b}}\,\vec{a} = \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{12}{3} = 4$, so

$d^2 = |\vec{a}|^2 - (\text{comp}_{\vec{b}}\,\vec{a})^2 = 65 - 16 = 49$ so $d = \boxed{7}$

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(10) $f(x) = (3x-1)^{-2}$ $a = 1$

$f'(x) = -2(3x-1)^{-3} \cdot 3$

$f''(x) = 6(3x-1)^{-4} \cdot 3^2$

$f'''(x) = -24(3x-1)^{-5} \cdot 3^3$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$	
0	1/4	1/4	
1	−3/4	−3/4	
2	27/8	27/16	← $\frac{1}{2} \cdot \frac{27}{8}$
3	−81/4	−27/8	← $\frac{1}{6}\left(-\frac{81}{4}\right)$

$\frac{1}{(3x-1)^2} = \boxed{\frac{1}{4} - \frac{3}{4}(x-1) + \frac{27}{16}(x-1)^2 - \frac{27}{8}(x-1)^3 + \cdots}$

(11)
$$e^{3x}\cos x = \left(1 + 3x + \frac{(3x)^2}{2!} + \frac{(3x)^3}{3!} + \cdots\right)\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$$
$$= \left(1 + 3x + \frac{9}{2}x^2 + \frac{9}{2}x^3 + \cdots\right)\left(1 - \frac{x^2}{2} + \frac{x^4}{24} - \cdots\right)$$
$$= 1 + 3x + \left(\frac{9}{2} - \frac{1}{2}\right)x^2 + \left(\frac{9}{2} - \frac{3}{2}\right)x^3 + \cdots$$
$$= \boxed{1 + 3x + 4x^2 + 3x^3 + \cdots}$$

(12) $\underline{\sum_{n=0}^{\infty} x^n} = 1 + x + x^2 + x^3 + \cdots = \underline{\frac{1}{1-x}}$, SO DIFFERENTIATING GIVES

$\underline{\sum_{n=1}^{\infty} n x^{n-1}} = \underline{\frac{1}{(1-x)^2}}$ AND MULTIPLYING BY X GIVES

$\underline{\sum_{n=1}^{\infty} n x^{n}} = \underline{\frac{x}{(1-x)^2}}$. DIFFERENTIATING AGAIN GIVES

$$\underline{\sum_{n=1}^{\infty} n^2 x^{n-1}} = \frac{(1-x)^2 \cdot 1 - x \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{1 - x + 2x}{(1-x)^3} = \underline{\frac{1+x}{(1-x)^3}}$$

THEN
$$\sum_{n=1}^{\infty} (-1)^{n+1}\frac{n^2}{3^{n-2}} = 3\sum_{n=1}^{\infty} (-1)^{n-1}\frac{n^2}{3^{n-1}} = 3\sum_{n=1}^{\infty} n^2\left(-\frac{1}{3}\right)^{n-1}$$
$$= 3\left[\frac{1 + \left(-\frac{1}{3}\right)}{\left(1 - \left(-\frac{1}{3}\right)\right)^3}\right] = 3 \cdot \frac{2/3}{64/27} = \boxed{\frac{27}{32}}$$