MATH 21C — ANSWERS – TEST 2

① $\vec{u} = \langle 2,1,2\rangle$ AND $\vec{v} = \langle 4,-4,7\rangle$

$$\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{u}|\,|\vec{v}|} = \frac{8-4+14}{\sqrt{9}\,\sqrt{81}} = \frac{18}{3\cdot 9} = \frac{2}{3}, \quad \text{so } \boxed{\theta = \cos^{-1}\frac{2}{3}}$$

② $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{(x-4)^n}{\sqrt{n}\,5^n}$

1) $$\ell = \lim_{n\to\infty} \frac{|u_{n+1}|}{|u_n|} = \lim_{n\to\infty} \frac{|x-4|^{n+1}}{\sqrt{n+1}\,5^{n+1}}\cdot\frac{\sqrt{n}\,5^n}{|x-4|^n} = \lim_{n\to\infty}\frac{\sqrt{n}}{\sqrt{n+1}}\cdot\frac{|x-4|}{5}$$
$$= \lim_{n\to\infty}\sqrt{\frac{n}{n+1}}\cdot\frac{|x-4|}{5} = 1\cdot\frac{|x-4|}{5} = \frac{|x-4|}{5}$$

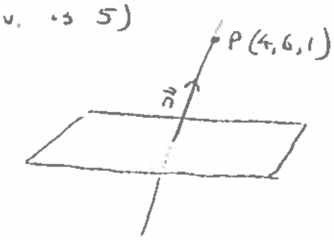
2) $\ell < 1$ IFF $\frac{|x-4|}{5} < 1$ IFF $|x-4| < 5$ IFF $-5 < x-4 < 5$ IFF $-1 < x < 9$.

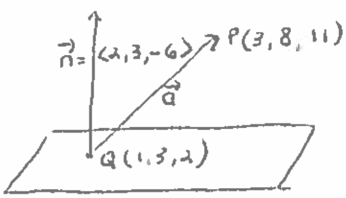
3) a) $x=-1$: $\displaystyle\sum_{n=1}^{\infty} (-1)^n\frac{(-5)^n}{\sqrt{n}\,5^n} = \sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ DIVERGES (P-SERIES, $P \le 1$)

b) $x=9$: $\displaystyle\sum_{n=1}^{\infty} (-1)^n\frac{5^n}{\sqrt{n}\,5^n} = \sum_{n=1}^{\infty}(-1)^n\frac{1}{\sqrt{n}}$ CONVERGES BY THE AST SINCE
1) $\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0$ AND 2) $\frac{1}{\sqrt{n}} \ge \frac{1}{\sqrt{n+1}}$ FOR ALL n,

INTERVAL OF CONVERGENCE: $\boxed{(-1,9]}$ (THE RADIUS OF CONV. IS 5)

③ SINCE THE LINE IS PERPENDICULAR TO THE PLANE,
$\vec{n} = \langle 3,-5,9\rangle$ IS A DIRECTION VECTOR FOR THE LINE;
SO $\boxed{x = 4+3t,\quad y = 6-5t,\quad z = 1+9t}$ ARE
PARAMETRIC EQUATIONS FOR THE LINE.



④ 

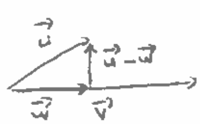
LET $\vec{a} = \overrightarrow{QP} = \langle 2,5,9\rangle$; THEN

$$D = \left|\text{comp}_{\vec{n}}\,\vec{a}\right| = \left|\frac{\vec{a}\cdot\vec{n}}{|\vec{n}|}\right| = \frac{|\vec{a}\cdot\vec{n}|}{|\vec{n}|} = \frac{|4+15-54|}{\sqrt{4+9+36}} = \frac{|-35|}{\sqrt{49}} = \frac{35}{7} = \boxed{5}$$

OR THE PLANE HAS EQUATION $2x+3y-6z=-1$, (SINCE $2(1)+3(3)-6(2) = -1$)

SO $$D = \frac{|2(3)+3(8)-6(11)+1|}{\sqrt{2^2+3^2+6^2}} = \frac{|-35|}{\sqrt{49}} = \frac{35}{7} = \boxed{5}$$

⑤ $\vec{u} = \langle 8,10,5\rangle$ AND $\vec{v} = \langle 5,1,2\rangle$



$$\vec{w} = \text{proj}_{\vec{v}}\,\vec{u} = \left(\frac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v} = \frac{60}{30}\vec{v} = 2\vec{v} = \boxed{\langle 10,2,4\rangle}$$

OR LET $\vec{w} = k\vec{v} = \langle 5k,k,2k\rangle$, SO $\vec{u}-\vec{w} = \langle 8-5k,\,10-k,\,5-2k\rangle$.

THEN $(\vec{u}-\vec{w})\cdot\vec{v} = 0 \implies 40-25k+10-k+10-4k = 0 \implies 30k = 60 \implies k=2$,

SO $\vec{w} = \boxed{\langle 10,2,4\rangle}$

MATH 21C ANSWERS – TEST 2

(6) $\sqrt{1+x} = (1+x)^{1/2} = 1 + \frac{1}{2}x + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{4!}x^4 + \cdots$

$= 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{1}{16}x^3 - \frac{5}{128}x^4 + \cdots$, so sub. x^3 for x gives

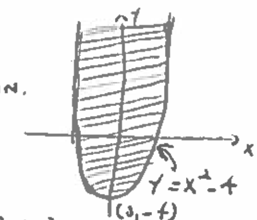
$$\int_0^{.2}\sqrt{1+x^3}\,dx = \int_0^{.2}\left(1 + \frac{1}{2}x^3 - \frac{1}{8}x^6 + \frac{1}{16}x^9 - \frac{5}{128}x^{12} + \cdots\right)dx$$

$$= \left[x + \frac{x^4}{8} - \frac{x^7}{56} + \frac{x^{10}}{160} - \frac{5}{13(128)}x^{13} + \cdots\right]_0^{.2}$$

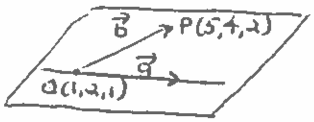
$$\approx \boxed{.2 + \frac{(.2)^4}{8} - \frac{(.2)^7}{56} + \frac{(.2)^{10}}{160}} \quad \text{with } |E| < \frac{5}{13(128)}(.2)^{13} \quad \left(\leftarrow \frac{5}{1664}(.2)^{13}\right)$$

(7) $f(x,y) = \sqrt{y+4-x^2}$ is defined where $y+4-x^2 \geq 0$ so $y \geq x^2 - 4$;

1) Boundary: $y = x^2 - 4$ 2) Test $(0,0)$: $0 \geq 0 - 4$, so $(0,0)$ is in the domain.



(8)



Let $\vec{a} = \langle 3,2,4\rangle$ and $\vec{b} = \vec{QP} = \langle 4,2,1\rangle$.

Then $\vec{n} = \vec{b}\times\vec{a} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 2 & 1 \\ 3 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 2 & 1 \\ 2 & 4 \end{vmatrix}\vec{i} - \begin{vmatrix} 4 & 1 \\ 3 & 4 \end{vmatrix}\vec{j} + \begin{vmatrix} 4 & 2 \\ 3 & 2 \end{vmatrix}\vec{k}$

$= \langle 6, -13, 2\rangle$ is a normal vector,

so $\boxed{6x - 13y + 2z = -18}$ is an equation of the plane (since $6(1) - 13(2) + 2(1) = -18$).

(9) a) $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, so $\sqrt[3]{e} = e^{1/3} = 1 + \frac{1}{3} + \frac{\left(\frac{1}{3}\right)^2}{2!} + \frac{\left(\frac{1}{3}\right)^3}{3!} + \cdots \approx \boxed{1 + \frac{1}{3} + \frac{1}{18} + \frac{1}{162}}$

b) $E = R_3\left(\frac{1}{3}\right) = \frac{f^{(4)}(c)}{4!}\left(\frac{1}{3} - 0\right)^4 = \frac{e^c}{24}\cdot\frac{1}{81}$ where $0 < c < 1/3$, so $e^c < e^{1/3}$ and

so $E < \frac{e^{1/3}}{24(81)} < \frac{7/5}{24(81)} = \frac{7}{120(81)} = \frac{7}{9720}$ (since $e < \left(\frac{7}{5}\right)^3$)

(10) $$\int_0^{.4}\frac{x^3}{2+x^5}\,dx = \int_0^{.4}\frac{x^3/2}{1+x^5/2}\,dx = \int_0^{.4}\frac{x^3/2}{1-\left(-x^5/2\right)}\,dx \quad \left(a = \frac{x^3}{2},\ r = -\frac{x^5}{2}\right)$$

$$= \int_0^{.4}\left(\frac{x^3}{2} - \frac{x^8}{4} + \frac{x^{13}}{8} - \frac{x^{18}}{16} + \cdots\right)dx$$

$$= \left[\frac{x^4}{8} - \frac{x^9}{36} + \frac{x^{14}}{112} - \frac{x^{19}}{19(16)} + \cdots\right]_0^{.4}$$

$$\approx \boxed{\frac{(.4)^4}{8} - \frac{(.4)^9}{36} + \frac{(.4)^{14}}{112} - \frac{(.4)^{19}}{304}}$$

MATH 21C ANSWERS – TEST 2

(11) APPROXIMATE $\sqrt{11}$ USING THE FIRST 4 TERMS OF THE TAYLOR SERIES FOR $f(x)=\sqrt{x}$ AT $a=9$.

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2}x^{-1/2}$

$f''(x) = -\frac{1}{4}x^{-3/2}$

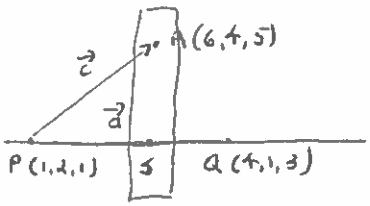
$f'''(x) = \frac{3}{8}x^{-5/2}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$	
0	3	3	
1	$\frac{1}{2}\cdot\frac{1}{3}$	$1/6$	
2	$-\frac{1}{4}\cdot\frac{1}{3^3}$	$-\frac{1}{8\cdot 27}$	$\longleftarrow \frac{1}{2}\left(-\frac{1}{4}\cdot\frac{1}{3^3}\right)$
3	$\frac{3}{8}\cdot\frac{1}{3^5}$	$\frac{1}{16\cdot 243}$	$\longleftarrow \frac{1}{6}\left(\frac{3}{8}\cdot\frac{1}{3^5}\right)$

$$\sqrt{x} = 3 + \frac{1}{6}(x-9) - \frac{1}{8(27)}(x-9)^2 + \frac{1}{16(243)}(x-9)^3 - \cdots, \quad \text{SO}$$

$$\sqrt{11} = 3 + \frac{1}{6}\cdot 2 - \frac{1}{8(27)}\cdot 2^2 + \frac{1}{16(243)}\cdot 2^3 - \cdots \approx \boxed{3 + \frac{1}{3} - \frac{1}{54} + \frac{1}{486}}$$

(12)



THE LINE THROUGH P AND Q HAS PARAMETRIC EQUATIONS

$x = 1+3t,\quad y = 2-t,\quad z = 1+2t$

SINCE $\vec{a} = \vec{PQ} = \langle 3,-1,2\rangle$ IS PARALLEL TO THE LINE.

THE PLANE THROUGH R WHICH IS PERPENDICULAR TO THE LINE HAS NORMAL VECTOR $\vec{n} = \vec{a} = \langle 3,-1,2\rangle$, SO IT HAS EQUATION $3x - y + 2z = 3(6) - 4 + 2(5)$ OR

$3x - y + 2z = 24$. THE LINE AND PLANE INTERSECT WHERE

$3(1+3t) - (2-t) + 2(1+2t) = 24$, $14t = 21$, $t = 3/2$, SO

$\boxed{x = \frac{11}{2},\quad y = \frac{1}{2},\quad z = 4}$

OR THE CLOSEST POINT S HAS THE PROPERTY THAT $\vec{RS}$ IS ORTHOGONAL TO $\vec{PQ}$;

SO IF $\vec{b} = \vec{RS} = \langle 3t-5, -2-t, -4+2t\rangle$, $\vec{b}\cdot\vec{a} = 0$ GIVES

$3(3t-5) - (-2-t) + 2(-4+2t) = 0$. THEN $14t = 21$, $t = 3/2$, AND

$S = \boxed{\left(\frac{11}{2}, \frac{1}{2}, 4\right)}$

OR LET S BE THE CLOSEST POINT, AND LET $\vec{c} = \vec{PR} = \langle 5,2,4\rangle$.

THEN $\vec{PS} = \text{PROJ}_{\vec{a}}\vec{c} = \left(\frac{\vec{c}\cdot\vec{a}}{\vec{a}\cdot\vec{a}}\right)\vec{a} = \frac{21}{14}\vec{a} = \frac{3}{2}\vec{a} = \left\langle \frac{9}{2}, -\frac{3}{2}, 3\right\rangle$,

SO $S = \left(1+\frac{9}{2},\ 2+\left(-\frac{3}{2}\right),\ 1+3\right) = \boxed{\left(\frac{11}{2}, \frac{1}{2}, 4\right)}$
