① $\underbrace{x^2 - 2xy + 3yz + z^2}_{g(x,y,z)} = 18$; $P(4,1,2)$

$\vec{\nabla} g = \langle 2x-2y,\ -2x+3z,\ 3y+2z \rangle$ so $\vec{\nabla} g(P) = \langle 6,-2,7 \rangle = \vec{n}$

$6x - 2y + 7z = d$ where $d = 6(4) - 2(1) + 7(2) = 36$: $\boxed{6x - 2y + 7z = 36}$

② $f(x,y,z) = 2xy + xz - 3yz$ at $P(1,1,4)$ in the direction of $\vec{v} = \langle 6,-3,-2 \rangle$

$\vec{\nabla} f = \langle 2y+z,\ 2x-3z,\ x-3y \rangle$ so $\vec{\nabla} f(P) = \langle 6,-10,-2 \rangle$

$\vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\langle 6,-3,-2 \rangle}{\sqrt{36+9+4}} = \left\langle \frac{6}{7}, -\frac{3}{7}, -\frac{2}{7} \right\rangle$

$D_{\vec{u}} f(P) = (\vec{\nabla} f(P)) \cdot \vec{u} = \dfrac{36+30+4}{7} = \dfrac{70}{7} = \boxed{10}$

③ max. and min. of $f(x,y) = -3x + 6y$ on the ellipse $\underbrace{x^2 + 4y^2}_{g(x,y)} = 200$

$-3 = \lambda(2x)$ so $\frac{1}{\lambda} = -\frac{2}{3}x$ $\quad -\frac{2}{3}x = \frac{4}{3}y$ so $x = -2y$

$6 = \lambda(8y)$ $\quad \frac{1}{\lambda} = \frac{4}{3}y$

$(-2y)^2 + 4y^2 = 200$ gives $4y^2 + 4y^2 = 200$, $8y^2 = 200$, $y^2 = 25$, $y = \pm 5$

$y = 5$: $f(-10,5) = 60$ is the max.

$y = -5$: $f(10,-5) = -60$ is the min.

④ $f(x,y,z) = x^2 e^y - 2x \ln z$; $P = (2, \ln 2, 1)$

$\vec{\nabla} f = \left\langle 2xe^y - 2\ln z,\ x^2 e^y,\ -\frac{2x}{z} \right\rangle$ so $\vec{\nabla} f(P) = \langle 8, 8, -4 \rangle = 4\langle 2,2,-1 \rangle$

a) $\max\limits_{|\vec{u}|=1} D_{\vec{u}} f(P) = |\vec{\nabla} f(P)| = 4\sqrt{2^2+2^2+1^2} = 4\sqrt{9} = \boxed{12}$

b) $\vec{u} = \dfrac{\vec{\nabla} f(P)}{|\vec{\nabla} f(P)|} = \dfrac{4\langle 2,2,-1 \rangle}{4 \cdot 3} = \boxed{\left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle}$

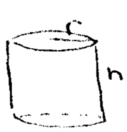
⑤ minimize $d^2 = (x-20)^2 + (y-6)^2 + (z-0)^2$ where $z = \sqrt{3x^2 + 2y^2}$,

so let $f(x,y) = (x-20)^2 + (y-6)^2 + 3x^2 + 2y^2$

$f_x = 2(x-20) + 6x = 0$ so $8x = 40$ $\boxed{x = 5}$

$f_y = 2(y-6) + 4y = 0$ so $6y = 12$ $\boxed{y = 2}$

Then $z = \sqrt{3(25) + 2(4)} = \boxed{\sqrt{83}}$, so $\boxed{(5, 2, \sqrt{83})}$ is the closest point on the cone.

⑥ 

$V = \pi r^2 h$, so $dV = V_r\, dr + V_h\, dh = 2\pi r h\, dr + \pi r^2\, dh$

$r = 5$, $dr = .09$ $\quad = 2\pi(5)(10)(.09) + \pi(5^2)(-.2)$

$h = 10$, $dh = -.2$ $\quad = 9\pi - 5\pi = \boxed{4\pi \text{ cm}^3}$

⑦ a) $f_x = 2x + 4y = 0$ so $2x = -4y$ $\quad x = -2y$

$f_y = 4x + 8y^3 = 0$

$-8y + 8y^3 = 0$ $\quad 8y(y^2 - 1) = 0$ so $y = 0$, $x = 0$ or $y^2 = 1$, $y = 1$, $x = -2$ or $y = -1$, $x = 2$

Critical points: $\boxed{(0,0),\ (-2,1),\ (2,-1)}$

(7) b) $f_{xx} = 2$, $f_{xy} = 4$, $f_{yy} = 24y^2$

	f_{xx}	f_{xy}	f_{yy}	D
$(0,0)$	2	4	0	-16
$(2,-1)$	2	4	24	32
$(-2,1)$	2	4	24	32

$\boxed{\text{SADDLE PT. AT } (0,0)}$
$\boxed{\text{LOCAL MIN. AT } (2,-1)}$
$\boxed{\text{LOCAL MIN. AT } (-2,1)}$

(8) $f(x,y,z) = 2x - 3y + z$ on ellipsoid $\underbrace{(x-5)^2 + 3y^2 + 2(z-3)^2}_{g(x,y,z)} = 30$

$2 = \lambda \cdot 2(x-5)$ $\qquad \frac{1}{\lambda} = x-5$ $\qquad x - 5 = -2y$ so $\underline{x = -2y+5}$

$-3 = \lambda \cdot 6y$ $\qquad \frac{1}{\lambda} = -2y$ $\qquad 4z - 12 = -2y$ so $4z = -2y + 12$, $\underline{z = -\frac{1}{2}y + 3}$

$1 = \lambda \cdot 4(z-3)$ $\qquad \frac{1}{\lambda} = 4z - 12$

Then $(-2y)^2 + 3y^2 + 2(-\frac{1}{2}y)^2 = 30$ so $4y^2 + 3y^2 + \frac{1}{2}y^2 = 30$,

$14y^2 + y^2 = 60$, $15y^2 = 60$, $y^2 = 4$, $\underline{y = \pm 2}$

$\underline{y=2}$: $f(1,2,2) = -2$ is the MIN.
$\underline{y=-2}$: $f(9,-2,4) = 28$ is the MAX.

(or use $\underline{x = 4z-7}$ and $\underline{y = -2z+6}$ to get $(4z-12)^2 + 3(-2z+6)^2 + 2(z-3)^2 = 30$,
$(4(z-3))^2 + 3((-2)(z-3))^2 + 2(z-3)^2 = 30$, $16(z-3)^2 + 12(z-3)^2 + 2(z-3)^2 = 30$,
$30(z-3)^2 = 30$, $(z-3)^2 = 1$, $z - 3 = \pm 1$ so $\underline{z=4}$ or $\underline{z=2}$)

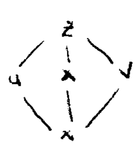
(9) $z = 3\ln(xy) + f(t)$ where $t = xy^2$ let $\underline{w = f(t)}$:

$\frac{\partial z}{\partial x} = 3 \cdot \frac{y}{xy} + \frac{dw}{dt} \cdot \frac{\partial t}{\partial x} = \frac{3}{x} + f'(t) \cdot y^2 = \underline{\frac{3}{x} + f'(xy^2) \cdot y^2}$

$\frac{\partial z}{\partial y} = 3 \cdot \frac{x}{xy} + \frac{dw}{dt} \cdot \frac{\partial t}{\partial y} = \frac{3}{y} + f'(t) \cdot 2xy = \underline{\frac{3}{y} + f'(xy^2) \cdot 2xy}$

so $2x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y} = 2x\left[\frac{3}{x} + f'(xy^2) \cdot y^2\right] - y\left[\frac{3}{y} + f'(xy^2) \cdot 2xy\right]$
$= 6 + 2xy^2 f'(xy^2) - 3 - 2xy^2 f'(xy^2) = \boxed{3}$

(10) $F(x) = \int_{x^3}^{x^4} e^{-xt^2}\,dt$ $\qquad$ let $z = F(x)$, $u = x^3$, $v = x^4$, so $\underline{z = \int_u^v e^{-xt^2}\,dt}$



$F'(x) = \frac{dz}{dx} = \frac{\partial z}{\partial u} \cdot \frac{du}{dx} + \left(\frac{\partial z}{\partial x}\right)_{u,v} \cdot \frac{dx}{dx} + \frac{\partial z}{\partial v} \cdot \frac{dv}{dx}$

$= -e^{-xu^2} \cdot 3x^2 + \int_u^v \frac{\partial}{\partial x}\left(e^{-xt^2}\right) dt \cdot 1 + e^{-xv^2} \cdot 4x^3$

$= \boxed{-3x^2 e^{-x^7} + \int_{x^3}^{x^4} -t^2 e^{-xt^2}\,dt + 4x^3 e^{-x^9}}$

(since $\frac{\partial z}{\partial u} = \frac{\partial}{\partial u}\left(-\int_v^u e^{-xt^2}\,dt\right) = -e^{-xu^2}$ and $\frac{\partial z}{\partial v} = \frac{\partial}{\partial v}\left(\int_u^v e^{-xt^2}\,dt\right) = e^{-xv^2}$)

(11) $f(x,y) = x^4 + 3xy^3 - 4x^3y + y^4$ at critical point $(0,0)$

1) on the x-axis,
$f(x,0) = x^4 > 0$ for $x \neq 0$, so $f(0,0) = 0$ is not a local MAX.

2) on the line $x = 2y$,
$f(2y,y) = 16y^4 + 6y^4 - 32y^4 + y^4 = -9y^4 < 0$ for $y \neq 0$,
so $f(0,0) = 0$ is not a local MIN.

Therefore $(0,0)$ is a $\boxed{\text{SADDLE POINT}}$.