① $x^2 - 2xy + 3yz + z^2 = 18$ AT $P(4,1,2)$

$\vec{\nabla F} = \langle 2x-2y, -2x+3z, 3y+2z \rangle$ SO $\vec{\nabla F}(P) = \langle 6,-2,7 \rangle$

$6x - 2y + 7z = d$ WHERE $d = 6(4) - 2(1) + 7(2) = 36$ SO

$\boxed{6x - 2y + 7z = 36}$

② $f(x,y) = 5x - 10$ ON ELLIPSE $\underbrace{x^2 + 4y^2}_{g(x,y)} = 128$

$5 = \lambda \cdot 2x \qquad \lambda = \frac{5}{2x} \qquad \frac{5}{2x} = -\frac{5}{4y}$ SO $2x = -4y$, $\underline{x = -2y}$

$-10 = \lambda \cdot 8y \qquad \lambda = -\frac{5}{4y}$

SUBSTITUTING INTO THE CONSTRAINT GIVES

$4y^2 + 4y^2 = 128$, $8y^2 = 128$, $y^2 = 16$, $\underline{y = \pm 4}$

$\underline{y=4}$: $\boxed{f(-8,4) = -80 \text{ IS THE MIN.}}$

$\underline{y=-4}$: $\boxed{f(8,-4) = 80 \text{ IS THE MAX.}}$

③ $f(x,y,z) = 2xy + xz + 3yz$ AT $P(2,1,1)$, $\vec{v} = \langle 6,3,2 \rangle$

$\vec{\nabla f} = \langle 2y+z, 2x+3z, x+3y \rangle$ SO $\vec{\nabla f}(P) = \langle 3,7,5 \rangle$

$\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{\langle 6,3,2 \rangle}{\sqrt{49}} = \left\langle \frac{6}{7}, \frac{3}{7}, \frac{2}{7} \right\rangle$

$\underline{D_{\vec{u}} f(P) = \vec{\nabla f}(P) \cdot \vec{u}} = \frac{18}{7} + \frac{21}{7} + \frac{10}{7} = \frac{49}{7} = \boxed{7}$

④ $f(x,y) = 18xy - 3xy^2 - x^2y$

$\underline{f_x = 18y - 3y^2 - 2xy = 0}$ $\qquad y(18 - 3y - 2x) = 0 \qquad \underline{y=0}$ OR $\underline{2x + 3y = 18}$

$\underline{f_y = 18x - 6xy - x^2 = 0}$ $\qquad x(18 - 6y - x) = 0 \qquad \underline{x=0}$ OR $\underline{x + 6y = 18}$

1) $y=0$, $x=0$: $\boxed{(0,0)}$

2) $y=0$, $x+6y=18$: $\boxed{(18,0)}$

3) $x=0$, $2x+3y=18$: $\boxed{(0,6)}$

4) $2x+3y=18$, $x+6y=18$: $\boxed{(6,2)}$

$$\begin{array}{r} 4x + 6y = 18 \\ - \quad x + 6y = 18 \\ \hline 3x = 18 \end{array}$$, $\underline{x=6}$, $\underline{y=2}$

⑤ $d^2 = (x-15)^2 + (y-12)^2 + (z-0)^2$ WHERE $z = \sqrt{2x^2 + 3y^2}$, SO LET

$\underline{f(x,y) = (x-15)^2 + (y-12)^2 + (2x^2 + 3y^2)}$

$f_x = 2(x-15) + 4x = 0$ GIVES $6x = 30$ SO $\boxed{x = 5}$

$f_y = 2(y-12) + 6y = 0$ GIVES $8y = 24$ SO $\boxed{y = 3}$

$\boxed{z = \sqrt{77}} \leftarrow \sqrt{2 \cdot 5^2 + 3 \cdot 3^2}$

(AT $(5,3)$, $f_{xx} = 6$, $f_{xy} = 0$, $f_{yy} = 8$, AND $D = 48$,
SO $x=5$ AND $y=3$ GIVES A MIN. SINCE $D > 0$ AND $f_{xx} > 0$)

⑥ $f(x,y,z) = x^2e^y - 2x\ln z$, $P = (2, \ln 2, 1)$

$\vec{\nabla} f = \langle 2xe^y - 2\ln z,\ x^2e^y,\ -\frac{2x}{z} \rangle$ so $\underline{\vec{\nabla} f(P) = \langle 8, 8, -4 \rangle} = 4\langle 2,2,-1\rangle$

a) $|\vec{\nabla} f(P)| = 4\,|\langle 2,2,-1\rangle| = 4\cdot 3 = \boxed{12}$

b) $\vec{u} = \frac{\vec{v}}{|\vec{v}|} = \frac{4\langle 2,2,-1\rangle}{4\cdot 3} = \boxed{\left\langle \frac{2}{3}, \frac{2}{3}, -\frac{1}{3} \right\rangle}$

⑦ $f(x,y) = x^2 + 8xy + 2y^4$

a) $f_x = 2x + 8y = 0$ so $\underline{x = -4y}$

$f_y = 8x + 8y^3 = 0$, $-32y + 8y^3 = 0$, $8y(y^2 - 4) = 0$, $\underline{y = 0}$ or $y^2 = 4$, $\underline{y = \pm 2}$

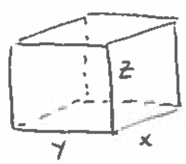
CRITICAL POINTS: $\boxed{(0,0),\ (-8,2),\ (8,-2)}$

b) $f_{xx} = 2$, $f_{xy} = 8$, $f_{yy} = 24y^2$

	f_{xx}	f_{xy}	f_{yy}	D
$(0,0)$	2	8	0	-64
$(-8,2)$	2	8	96	128
$(8,-2)$	2	8	96	128

SADDLE PT. AT $(0,0)$
LOCAL MIN. AT $(-8,2)$
LOCAL MIN. AT $(8,-2)$

⑧



(OPEN TOP)

MAXIMIZE $\underline{V = xyz}$, SUBJECT TO $\underbrace{xy + 2yz + 2xz}_{g(x,y,z)} = 300$

$yz = \lambda(y + 2z)$ $\quad \frac{1}{\lambda} = \frac{y+2z}{yz} = \frac{1}{z} + \frac{2}{y}$

$xz = \lambda(x + 2z)$ $\quad \frac{1}{\lambda} = \frac{x+2z}{xz} = \frac{1}{z} + \frac{2}{x}$

$xy = \lambda(2y + 2x)$ $\quad \frac{1}{\lambda} = \frac{2y+2x}{xy} = \frac{2}{x} + \frac{2}{y}$

so $\frac{1}{z} + \frac{2}{y} = \frac{1}{z} + \frac{2}{x}$, $\frac{2}{y} = \frac{2}{x}$, $\underline{x = y}$ (USING $x, y, z > 0$ so $\lambda > 0$)

AND $\frac{1}{z} + \frac{2}{y} = \frac{2}{x} + \frac{2}{y}$, $\frac{1}{z} = \frac{2}{x}$, $\underline{x = 2z}$

THEN SUBSTITUTING INTO THE CONSTRAINT GIVES

$x^2 + x^2 + x^2 = 300$, $3x^2 = 300$, $x^2 = 100$, $\underline{x = 10}$ (SINCE $x > 0$)

THEN $\underline{y = 10}$ AND $\underline{z = 5}$, so $V = xyz = \boxed{500\ \text{cm}^3}$

OR use $\underline{xyz} = \lambda(xy + 2xz)$ (MULT. BY x)

$xyz = \lambda(xy + 2yz)$ (MULT. BY y)

$\underline{xyz} = \lambda(2yz + 2xz)$ (MULT. BY z)

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(9) $h(t) = g(t^3 - 2t^2,\ t^2 + 8t)$

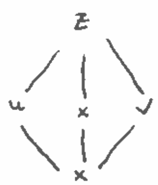
LET $z = h(t)$, $x = t^3 - 2t^2$, $y = t^2 + 8t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}, \quad \text{so}$$

$$h'(t) = g_x(x,y)\cdot(3t^2 - 4t) + g_y(x,y)\cdot(2t+8) \quad \text{AND}$$

$$h'(3) = \boxed{15 g_x(9,33) + 14 g_y(9,33)}$$

(10) $F(x) = \displaystyle\int_{x^2}^{x^5} e^{-x^2 t^3}\,dt$ LET $u = x^2$, $v = x^5$, $z = F(x)$:



$$\frac{dz}{dx} = \frac{\partial z}{\partial u}\cdot\frac{du}{dx} + \frac{\partial z}{\partial v}\cdot\frac{dv}{dx} + \left(\frac{\partial z}{\partial x}\right)_{u,v}\cdot\frac{dx}{dx}$$

$$= \frac{\partial}{\partial u}\left(\int_u^v e^{-x^2 t^3}\,dt\right)\cdot 2x + \frac{\partial}{\partial v}\left(\int_u^v e^{-x^2 t^3}\,dt\right)\cdot 5x^4 + \int_{x^2}^{x^5} \frac{\partial}{\partial x}\left(e^{-x^2 t^3}\right)dt\cdot 1$$

$$\frac{\partial}{\partial u}\left(\int_u^v e^{-x^2t^3}\,dt\right) = \frac{\partial}{\partial u}\left(-\int_v^u e^{-x^2t^3}\,dt\right)$$

$$= -e^{-x^2u^3}\cdot 2x + e^{-x^2v^3}\cdot 5x^4 + \int_{x^2}^{x^5} e^{-x^2t^3}(-2x t^3)\,dt$$

$$= \boxed{-2x e^{-x^8} + 5x^4 e^{-x^{17}} - 2x\int_{x^2}^{x^5} e^{-x^2t^3}\cdot t^3\,dt}$$

(11) $f(x,y) = x^4 + 3xy^3 - 4x^3y + y^4$ AT $(0,0)$

1) ON THE X-AXIS,

$f(x,0) = x^4 > 0$ FOR $x \neq 0$, SO $f(0,0) = 0$ IS NOT A LOCAL MAX.

2) ON THE LINE $x = 2y$,

$f(2y,y) = 16y^4 + 6y^4 - 32y^4 + y^4 = -9y^4 < 0$ FOR $y \neq 0$,

SO $f(0,0) = 0$ IS NOT A LOCAL MIN.

THEREFORE $(0,0)$ IS A $\boxed{\text{SADDLE POINT}}$