MATH 21C Sec. 10.8 - EXAMPLES

A) FIND THE TAYLOR SERIES FOR $f(x) = \frac{1}{3x-10}$ CENTERED AT $a=4$.

$f(x) = (3x-10)^{-1}$

$f'(x) = -(3x-10)^{-2} \cdot 3$

$f''(x) = 2(3x-10)^{-3} \cdot 3^2$

$f'''(x) = -6(3x-10)^{-4} \cdot 3^3$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$	
0	$1/2$	$1/2$	
1	$-3/4$	$-3/4$	
2	$18/8$	$9/8$	$\leftarrow \left(\frac{18}{8} \cdot \frac{1}{2}\right)$
3	$-6 \cdot 27/16$	$-27/16$	$\leftarrow \left(-\frac{6}{16} \cdot 27 \cdot \frac{1}{6}\right)$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$= \boxed{\frac{1}{2} - \frac{3}{4}(x-4) + \frac{9}{8}(x-4)^2 - \frac{27}{16}(x-4)^3 + \cdots}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{3^n}{2^{n+1}} (x-4)^n}$$

B) FIND THE TAYLOR SERIES FOR $f(x) = \sqrt{x}$ CENTERED AT $a=9$.

$f(x) = x^{1/2}$

$f'(x) = \frac{1}{2} x^{-1/2}$

$f''(x) = -\frac{1}{4} x^{-3/2}$

$f'''(x) = \frac{3}{8} x^{-5/2}$

$f^{(4)}(x) = -\frac{15}{16} x^{-7/2}$

n	$f^{(n)}(a)$	$\frac{f^{(n)}(a)}{n!}$	
0	3	3	
1	$\frac{1}{2} \cdot \frac{1}{3}$	$1/6$	
2	$-\frac{1}{4} \cdot \frac{1}{3^3}$	$-1/216$	$\leftarrow \left(-\frac{1}{4 \cdot 27} \cdot \frac{1}{2}\right)$
3	$\frac{3}{8} \cdot \frac{1}{3^5}$	$\frac{1}{3888}$	$\leftarrow \left(\frac{3}{8 \cdot 3^5} \cdot \frac{1}{6}\right)$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

$$= \boxed{3 + \frac{1}{6}(x-9) - \frac{1}{216}(x-9)^2 + \frac{1}{3888}(x-9)^3 - \cdots}$$

$$= \boxed{3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1 \cdot 3 \cdot 5 \cdot 7 \cdots (2n-3)}{2^n 3^{2n-1} n!} (x-9)^n} \; *$$

(USING THE CONVENTION THAT AN "EMPTY PRODUCT" IS 1)

$$= \boxed{3 + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(2n-2)!}{6^{2n-1} n! (n-1)!} (x-9)^n}$$

(MULTIPLYING BY $\frac{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n-2)}{2 \cdot 4 \cdot 6 \cdot 8 \cdots (2n-2)}$)

*REMARK NOTICE THAT $f''(x) = \frac{1}{2}\left(-\frac{1}{2}\right) x^{-3/2}$, $f'''(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right) x^{-5/2}$,

$f^{(4)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right) x^{-7/2}$,

$f^{(5)}(x) = \frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\left(-\frac{7}{2}\right) x^{-9/2}$, ...

SEC. 10.8 – FINDING TAYLOR SERIES USING SUBSTITUTION

(A) FIND THE MACLAURIN SERIES FOR $f(x) = e^{-x^3}$.

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots, \quad \text{so}$$

$$e^{-x^3} = 1 + (-x^3) + \frac{(-x^3)^2}{2!} + \frac{(-x^3)^3}{3!} + \frac{(-x^3)^4}{4!} + \cdots$$

$$= \boxed{1 - x^3 + \frac{x^6}{2!} - \frac{x^9}{3!} + \frac{x^{12}}{4!} - \cdots} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{3n}}{n!}}$$

(B) FIND THE MACLAURIN SERIES FOR $f(x) = \ln(1 + x^5)$.

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots, \quad \text{so}$$

$$\ln(1+x^5) = x^5 - \frac{(x^5)^2}{2} + \frac{(x^5)^3}{3} - \frac{(x^5)^4}{4} + \cdots$$

$$= \boxed{x^5 - \frac{x^{10}}{2} + \frac{x^{15}}{3} - \frac{x^{20}}{4} + \cdots} = \boxed{\sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{5n}}{n}}$$

(C) FIND THE MACLAURIN SERIES FOR $f(x) = \dfrac{x}{3 + x^2}$.

$$\frac{x}{3+x^2} = \frac{x/3}{1 + x^2/3} = \frac{x/3}{1 - \left(-\frac{x^2}{3}\right)} \longleftarrow \frac{a}{1-r}, \quad \text{WHERE } \underline{a = \tfrac{x}{3}} \text{ AND } \underline{r = -\tfrac{x^2}{3}}!$$

$$= \boxed{\frac{x}{3} - \frac{x^3}{9} + \frac{x^5}{27} - \frac{x^7}{81} + \cdots} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{3^{n+1}}} \longleftarrow (ar^n)$$

(D) FIND THE TAYLOR SERIES FOR $f(x) = \dfrac{4}{2+3x}$ CENTERED AT $a = 1$.

$$\frac{4}{2+3x} = \frac{4}{5 + 3(x-1)} = \frac{4/5}{1 + \frac{3}{5}(x-1)} = \frac{4/5}{1 - \left(-\frac{3}{5}(x-1)\right)} \longleftarrow \frac{a}{1-r}, \quad \text{WHERE } \underline{a = \tfrac{4}{5}} \text{ AND } \underline{r = -\tfrac{3}{5}(x-1)}$$

$$= \boxed{\frac{4}{5} - \frac{12}{25}(x-1) + \frac{36}{125}(x-1)^2 - \frac{108}{625}(x-1)^3 + \cdots}$$

$$= \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{4(3^n)}{5^{n+1}} (x-1)^n} \longleftarrow (ar^n)$$

(E) FIND THE MACLAURIN SERIES FOR $f(x) = \sin(x^2)$.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots, \quad \text{so}$$

$$\sin(x^2) = (x^2) - \frac{(x^2)^3}{3!} + \frac{(x^2)^5}{5!} - \frac{(x^2)^7}{7!} + \cdots$$

$$= \boxed{x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} - \frac{x^{14}}{7!} + \cdots} = \boxed{\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+2}}{(2n+1)!}}$$

(I) As we've seen, a power series $\sum_{n=0}^{\infty} c_n (x-a)^n$ can be differentiated or integrated term-by-term inside its interval of convergence, and the resulting series will have the same radius of convergence (but may behave differently at endpoints).

In addition, we can say that

1) If a power series diverges at an endpoint of its interval of convergence, then the derivative of the power series will diverge at this endpoint also.

2) If a power series converges at an endpoint of its interval of convergence, then the integral of the power series will converge at this endpoint also.

(II) Let $\sum_{n=0}^{\infty} c_n x^n$ be a power series centered at 0 with positive radius of convergence r. If the power series converges at an endpoint (r or $-r$), then it has 1-sided continuity at that endpoint.

Ex As in Ex. 6, we saw that

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \cdots \quad \text{for } -1 < x < 1$$

by integrating the series for $\frac{1}{1+x}$ at $a = 0$.

Notice that this series also converges for $x = 1$ by the AST, so it is continuous from the left at 1.

Therefore $\lim_{x \to 1^-} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$,

so $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n}$

$$= \lim_{x \to 1^-} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^n}{n}$$

$$= \lim_{x \to 1^-} \ln(1+x) = \ln 2.$$