Math 21C **Classifying Infinite Series**

We can classify any infinite series $\sum_{n=1}^{\infty} a_n$ into one of 3 categories:

a) Absolutely Convergent – $\sum_{n=1}^{\infty} |a_n|$ converges;

b) Conditionally Convergent – $\sum_{n=1}^{\infty} |a_n|$ diverges, but $\sum_{n=1}^{\infty} a_n$ converges;

c) Divergent – $\sum_{n=1}^{\infty} a_n$ diverges.

<u>Remark</u> Notice that conditional convergence is not possible for a positive-term series.

One important distinction between an absolutely convergent series and a conditionally convergent series is the fact that the terms of an absolutely convergent series can be rearranged in any order without affecting the sum of the series, whereas the terms of a conditionally convergent series can be rearranged so that the series converges to any specified number or even diverges.

To classify the series $\sum_{n=1}^{\infty} a_n$, we can use the following procedure:

1 Check to see that $\lim_{n\to\infty} a_n = 0$.

If you can show that $\lim_{n\to\infty} a_n \neq 0$, then you know that $\sum_{n=1}^{\infty} a_n$ diverges by the nth Term Test for Divergence (or, for short, the Divergence Test). [If you can see that $\lim_{n\to\infty} a_n = 0$, or if you are not sure, then proceed to the next step.]

2 Test for absolute convergence by seeing whether $\sum_{n=1}^{\infty} |a_n|$ converges. Since $\sum_{n=1}^{\infty} |a_n|$ is a positive-term series, we have several tests at our disposal to use here:

a) the Comparison Test

b) the Limit Comparison Test

c) the Ratio Test

d) the Root Test

e) the Integral Test

Sometimes there is more than one test which we could apply; the following are some general guidelines we can use in choosing a test:

• If the series $\sum_{n=1}^{\infty} |a_n|$ involves a factorial, then the Ratio Test is probably the best choice.

• If the terms of the series $\sum_{n=1}^{\infty} |a_n|$ are quotients of two polynomials in n (or in some root of n), such as $\sum_{n=1}^{\infty} \frac{n+5}{n^2+n^3}$ or $\sum_{n=1}^{\infty} \frac{\sqrt{n}+2}{n+6\sqrt[3]{n}}$, try using the Comparison Test or Limit Comparison Test.

• If the series $\sum_{n=1}^{\infty} |a_n|$ involves an exponential, such as 2^n or n^n, then try using either the Ratio Test, the Comparison Test or Limit Comparison Test, or the Root Test. (In particular, use the Root Test if $|a_n|$ is a quotient of two exponentials or is an exponential itself.)

• If replacing n by x in $|a_n|$ gives a decreasing function which is easy to integrate, try using the Integral Test.

• If the series $\sum_{n=1}^{\infty} |a_n|$ involves a sine or cosine function, try showing that $\sum_{n=1}^{\infty} |a_n|$ converges using the Comparison Test (assuming that you have already checked that the Divergence Test does not apply).

• • If $\sum_{n=1}^{\infty} |a_n|$ converges by any of the 5 tests we have discussed, then $\sum_{n=1}^{\infty} a_n$ converges absolutely and we are done.

• • If $\sum_{n=1}^{\infty} |a_n|$ diverges by the Ratio Test or the Root Test, then $\sum_{n=1}^{\infty} a_n$ diverges (by the Divergence Test, since in fact $\lim_{n\to\infty} a_n \neq 0$) and we are done.

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3 If $\sum_{n=1}^{\infty} |a_n|$ diverges by either the Comparison Test, Limit Comparison Test, or Integral Test, then test $\sum_{n=1}^{\infty} a_n$ for conditional convergence using the Alternating Series Test (if it applies).