MATH 21C   Sec. 10.6 – REARRANGING SERIES

TH 1 LET $\sum_{n=1}^{\infty} a_n$ BE A POSITIVE-TERM SERIES WHICH CONVERGES. IF $\sum_{n=1}^{\infty} b_n$ IS ANY REARRANGEMENT OF $\sum_{n=1}^{\infty} a_n$, THEN $\sum_{n=1}^{\infty} b_n$ CONVERGES AND $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

PF LET $\{S_n\}$ AND $\{T_n\}$ BE THE SEQUENCES OF PARTIAL SUMS FOR $\sum_{n=1}^{\infty} a_n$ AND $\sum_{n=1}^{\infty} b_n$, RESPECTIVELY, AND LET $\sum_{n=1}^{\infty} a_n = S$.

THEN $T_n = b_1 + \cdots + b_n = a_{i_1} + \cdots + a_{i_n} \le S_N$ WHERE $N = \max\{i_1, \ldots, i_n\}$,

SO $T_n \le S_N \le S$ FOR ALL n. THEREFORE $\{T_n\}$ CONVERGES SINCE IT IS INCREASING AND BOUNDED ABOVE, SO $\lim_{n\to\infty} T_n = T$ WHERE $T \le S$.

THEREFORE $\sum_{n=1}^{\infty} b_n$ CONVERGES, AND $\sum_{n=1}^{\infty} b_n \le \sum_{n=1}^{\infty} a_n$. SINCE $\sum_{n=1}^{\infty} a_n$ IS ALSO A REARRANGEMENT OF $\sum_{n=1}^{\infty} b_n$, WE HAVE $\sum_{n=1}^{\infty} a_n \le \sum_{n=1}^{\infty} b_n$; SO $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

WE CAN GENERALIZE TH. 1 AS FOLLOWS:

TH 2 LET $\sum_{n=1}^{\infty} a_n$ BE AN ABSOLUTELY CONVERGENT SERIES. IF $\sum_{n=1}^{\infty} b_n$ IS ANY REARRANGEMENT OF $\sum_{n=1}^{\infty} a_n$, THEN $\sum_{n=1}^{\infty} b_n$ IS ABSOLUTELY CONVERGENT AND $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n$.

REMARK THE FACT THAT $\sum_{n=1}^{\infty} b_n$ IS ABSOLUTELY CONVERGENT FOLLOWS FROM TH. 1.

EX $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots + (-1)^{n+1}\cdot\frac{1}{2^{n-1}} + \cdots$ IS ABSOLUTELY CONVERGENT

WITH $S = \frac{a}{1-r} = \frac{1}{1-(-1/2)} = \frac{2}{3}$; SO

$1 + \frac{1}{4} - \frac{1}{2} + \frac{1}{16} - \frac{1}{8} + \frac{1}{64} - \frac{1}{32} + \cdots$ IS ABSOLUTELY CONVERGENT WITH $S = \frac{2}{3}$.

TH 3 LET $\sum_{n=1}^{\infty} a_n$ BE A CONDITIONALLY CONVERGENT SERIES. THEN

a) FOR ANY GIVEN NUMBER S, THERE IS A REARRANGEMENT OF $\sum_{n=1}^{\infty} a_n$ WHICH HAS SUM EQUAL TO S.

b) THERE IS A REARRANGEMENT OF $\sum_{n=1}^{\infty} a_n$ WHICH DIVERGES.

EX THE ALTERNATING HARMONIC SERIES

$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$ IS CONDITIONALLY CONVERGENT WITH $\underline{S = \ln 2}$,

BUT THE REARRANGEMENT

$1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots$ HAS $\underline{S = \frac{3}{2}\ln 2}$;

$$\underline{1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots = \ln 2}$$

SO $\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \frac{1}{16} + \cdots = \frac{1}{2}\ln 2$

SO $\underline{0 + \frac{1}{2} + 0 - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \cdots = \frac{1}{2}\ln 2}$

ADDING THESE TWO CONVERGENT SERIES TERM-BY-TERM, AND OMITTING 0 TERMS, GIVES

$$\underline{1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \cdots = \frac{3}{2}\ln 2}$$