**TH 1** Let \( \sum_{n=1}^{\infty} a_n \) be a positive-term series which converges. If \( \sum_{n=1}^{\infty} b_n \) is any rearrangement of \( \sum_{n=1}^{\infty} a_n \), then \( \sum_{n=1}^{\infty} b_n \) converges and \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n \).

**PF** Let \( \{S_n\} \) and \( \{T_n\} \) be the sequences of partial sums for \( \sum_{n=1}^{\infty} a_n \) and \( \sum_{n=1}^{\infty} b_n \), respectively, and let \( \sum_{n=1}^{\infty} a_n = S \), then \( T_n = b_1 + \ldots + b_n = a_1 + \ldots + a_n \leq S_n \) where \( N = \max \{1, \ldots, |n|\} \), so \( T_n \leq S_n \leq S \) for all \( n \). Therefore \( \{T_n\} \) converges since it is increasing and bounded above, so \( \lim_{n \to \infty} T_n = T \) where \( T \leq S \).

Therefore \( \sum_{n=1}^{\infty} b_n \) converges, and \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n \). Since \( \sum_{n=1}^{\infty} a_n \) is also a rearrangement of \( \sum_{n=1}^{\infty} b_n \), we have \( \sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} b_n \).

We can generalize TH. 1 as follows:

**TH 2** Let \( \sum_{n=1}^{\infty} a_n \) be an absolutely convergent series. If \( \sum_{n=1}^{\infty} b_n \) is any rearrangement of \( \sum_{n=1}^{\infty} a_n \), then \( \sum_{n=1}^{\infty} b_n \) is absolutely convergent and \( \sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} a_n \).

**REMARK** The fact that \( \sum_{n=1}^{\infty} b_n \) is absolutely convergent follows from TH. 1.

**EX**

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots + (-1)^{n+1} \frac{1}{2^{n-1}} + \ldots \] is absolutely convergent

with \( S = \frac{1}{1 - (-\frac{1}{2})} = \frac{2}{3} \), so

\[ 1 + \frac{1}{2} - \frac{1}{3} + \frac{1}{4} - \frac{1}{5} + \frac{1}{6} - \frac{1}{7} + \ldots \] is absolutely convergent with \( S = \frac{3}{2} \).

**TH 3** Let \( \sum_{n=1}^{\infty} a_n \) be a conditionally convergent series, then

a) For any given number \( S \), there is a rearrangement of \( \sum_{n=1}^{\infty} a_n \) which has sum equal to \( S \).

b) There is a rearrangement of \( \sum_{n=1}^{\infty} a_n \) which diverges.

**EX**

The alternating harmonic series

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \ldots \] is conditionally convergent with \( S = \text{LN}2 \),

but the rearrangement

\[ 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} - \frac{1}{6} + \ldots \]

has \( S = \frac{3}{2} \text{LN}2 \);

\[ 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \ldots = \text{LN}2 \]

so

\[ \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} + \frac{1}{12} - \frac{1}{14} - \frac{1}{16} + \ldots = \frac{1}{2} \text{LN}2 \]

so

\[ 0 + \frac{1}{2} - \frac{1}{4} + 0 + \frac{1}{6} + 0 - \frac{1}{8} + \ldots = \frac{1}{2} \text{LN}2 \]

Adding these two convergent series term-by-term, and omitting \( 0 \) terms, gives

\[ 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \ldots = \frac{3}{2} \text{LN}2 \]