MATH 21C

SEC. 10.5 – SOLUTIONS

(37) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{(2n+1)!}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{n!} = \lim_{n\to\infty} \frac{n+1}{(2n+3)(2n+2)}$$

$$= \lim_{n\to\infty} \frac{1}{2(2n+3)} = 0 < 1,$$

so THE SERIES $\boxed{\text{CONVERGES}}$ BY THE RATIO TEST.

(38) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{n^n}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} = \lim_{n\to\infty} \frac{(n+1)n^n}{(n+1)^{n+1}}$$

$$= \lim_{n\to\infty} \frac{n^n}{(n+1)^n} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^n = \lim_{n\to\infty} \left(\frac{1}{1+\frac{1}{n}}\right)^n = \lim_{n\to\infty} \frac{1}{(1+\frac{1}{n})^n} = \frac{1}{e} < 1,$$

so THE SERIES $\boxed{\text{CONVERGES}}$ BY THE RATIO TEST.

(39) $\displaystyle\sum_{n=2}^{\infty} \frac{n}{(\ln n)^n}$

$$\rho = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{\ln n} = 0 < 1 \quad \text{(SINCE } n^{1/n} \to 1 \text{ AND } \ln n \to \infty\text{)},$$

so THE SERIES $\boxed{\text{CONVERGES}}$ BY THE ROOT TEST.

(42) $\displaystyle\sum_{n=1}^{\infty} \frac{3^n}{n^3 2^n}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)^3 2^{n+1}} \cdot \frac{n^3 2^n}{3^n} = \lim_{n\to\infty} \frac{3}{2} \cdot \left(\frac{n}{n+1}\right)^3 = \frac{3}{2} \cdot 1 = \frac{3}{2} > 1,$$

so THE SERIES $\boxed{\text{DIVERGES}}$ BY THE RATIO TEST.

OR
$$\rho = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \frac{3}{n^{3/n} \cdot 2} = \lim_{n\to\infty} \frac{3}{2} \cdot \frac{1}{(n^{1/n})^3} = \frac{3}{2} \cdot \frac{1}{1} = \frac{3}{2} > 1$$

(SINCE $n^{1/n} \to 1$), so THE SERIES $\boxed{\text{DIVERGES}}$ BY THE ROOT TEST.

---

(43) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{((n+1)!)^2}{(2n+2)!} \cdot \frac{(2n)!}{(n!)^2} = \lim_{n\to\infty} \left(\frac{(n+1)!}{n!}\right)^2 \cdot \frac{(2n)!}{(2n+2)!}$$

$$= \lim_{n\to\infty} (n+1)^2 \cdot \frac{1}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{(n+1)(n+1)}{(2n+2)(2n+1)} = \lim_{n\to\infty} \frac{n+1}{4n+2} = \frac{1}{4} < 1,$$

so THE SERIES $\boxed{\text{CONVERGES}}$ BY THE RATIO TEST.

(59) $\displaystyle\sum_{n=1}^{\infty} \frac{n^n}{2^{n^2}}$

$$\rho = \lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\frac{n^n}{2^{n^2}}\right)^{1/n} = \lim_{n\to\infty} \frac{(n^n)^{1/n}}{(2^{n^2})^{1/n}}$$

$$= \lim_{n\to\infty} \frac{n}{2^n} = \lim_{x\to\infty} \frac{x}{2^x} \underset{(\frac{\infty}{\infty})}{=} \lim_{x\to\infty} \frac{1}{2^x \ln 2} = 0 < 1,$$

so THE SERIES $\boxed{\text{CONVERGES}}$ BY THE ROOT TEST.

OR
$$\rho = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)^{n+1}}{2^{(n+1)^2}} \cdot \frac{2^{n^2}}{n^n} = \lim_{n\to\infty} \frac{(n+1)(n+1)^n}{2^{n^2+2n+1}} \cdot \frac{2^{n^2}}{n^n}$$

$$= \lim_{n\to\infty} \frac{(n+1)^n}{n^n} \cdot \frac{n+1}{2^{2n+1}} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^n \cdot \lim_{x\to\infty} \frac{x+1}{2^{2x+1}} \longleftarrow \left(\frac{\infty}{\infty}\right)$$

$$= \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n \cdot \lim_{x\to\infty} \frac{1}{2^{2x+1} \cdot 2\ln 2} = e \cdot 0 = 0 < 1,$$

so THE SERIES $\boxed{\text{CONVERGES}}$ BY THE RATIO TEST

(10) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{\ln n}$ $\boxed{\text{CONVERGES}}$ BY THE AST SINCE (IT IS CONDITIONALLY CONVERGENT.)

1) $\lim_{n\to\infty} \frac{1}{\ln n} = 0$ AND 2) $\frac{1}{\ln n} \geq \frac{1}{\ln(n+1)}$ FOR $n \geq 2$

---

(19) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n}{n^3+1}$

(A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{n}{n^3+1}$ COMPARE TO $\sum_{n=1}^{\infty} \frac{1}{n^2}$, WHICH CONVERGES SINCE IT IS A P-SERIES WITH $P>1$:

$\frac{n}{n^3+1} < \frac{n}{n^3} = \frac{1}{n^2}$ FOR ALL n,

SO $\sum_{n=1}^{\infty} |a_n|$ CONVERGES BY THE COMPARISON TEST AND THUS $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

(25) $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1+n}{n^2}$

(A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1+n}{n^2} = \sum_{n=1}^{\infty} \left(\frac{1}{n^2} + \frac{1}{n}\right)$ DIVERGES SINCE

1) $\sum_{n=1}^{\infty} \frac{1}{n^2}$ CONVERGES (P-SERIES, $P>1$) AND 2) $\sum_{n=1}^{\infty} \frac{1}{n}$ DIVERGES (HARMONIC SERIES)

(OR COMPARE $\sum_{n=1}^{\infty} |a_n|$ TO THE HARMONIC SERIES).

(B) SINCE 1) $\lim_{n\to\infty} \frac{1+n}{n^2} = \lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{1}{n}\right) = 0$

AND 2) $\frac{1}{n^2} + \frac{1}{n} \geq \frac{1}{(n+1)^2} + \frac{1}{n+1}$ FOR ALL n,

$\sum_{n=1}^{\infty} a_n$ CONVERGES BY THE AST; SO $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES CONDITIONALLY}}$.

(28) $\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$

(A) $\sum_{n=2}^{\infty} |a_n| = \sum_{n=2}^{\infty} \frac{1}{n \ln n}$ $f(x) = \frac{1}{x \ln x}$ IS CONT. AND DECREASING FOR $x \geq 2$,

AND $\int_2^{\infty} \frac{1}{x \ln x}\,dx = \lim_{T\to\infty} \int_2^{T} \frac{1}{x \ln x}\,dx$ (LET $u = \ln x$, $du = \frac{1}{x}\,dx$)

$= \lim_{T\to\infty} \int_{x=2}^{x=T} \frac{1}{u}\,du = \lim_{T\to\infty} \left[\ln u\right]_{x=2}^{x=T} = \lim_{T\to\infty} \left[\ln(\ln x)\right]_2^T$

$= \lim_{T\to\infty} \left(\ln(\ln T) - \ln(\ln 2)\right) = \infty$,

SO $\sum_{n=2}^{\infty} |a_n|$ DIVERGES BY THE INTEGRAL TEST.

(B) SINCE 1) $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$ AND 2) $\frac{1}{n \ln n} \geq \frac{1}{(n+1)\ln(n+1)}$ FOR $n \geq 2$,

$\sum_{n=2}^{\infty} (-1)^{n+1} \frac{1}{n \ln n}$ CONVERGES BY THE AST AND THEREFORE IT $\boxed{\text{CONVERGES CONDITIONALLY}}$.

(17) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ DIVERGES SINCE IT IS A P-SERIES WITH $P \le 1$.

B) $\sum_{n=1}^{\infty} (-1)^n \frac{1}{\sqrt{n}}$ CONVERGES BY THE AST SINCE 1) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ AND 2) $\frac{1}{\sqrt{n}} \ge \frac{1}{\sqrt{n+1}}$ FOR ALL n,

SO THE SERIES $\boxed{\text{CONVERGES CONDITIONALLY}}$.

(27) $\sum_{n=1}^{\infty} (-1)^n n^2 \left(\frac{2}{3}\right)^n$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} n^2 \left(\frac{2}{3}\right)^n$

SINCE $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{(n+1)^2 \left(\frac{2}{3}\right)^{n+1}}{n^2 \left(\frac{2}{3}\right)^n} = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 \cdot \frac{2}{3} = 1^2 \cdot \frac{2}{3} = \frac{2}{3} < 1$,

$\sum_{n=1}^{\infty} |a_n|$ CONVERGES BY THE RATIO TEST, SO $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

[OR SHOW THAT $\sum_{n=1}^{\infty} |a_n|$ CONVERGES USING THE ROOT TEST, AND THE FACT THAT $\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \left(n^{2/n} \cdot \frac{2}{3}\right) = \lim_{n\to\infty} (n^{1/n})^2 \cdot \frac{2}{3} = 1^2 \cdot \frac{2}{3} = \frac{2}{3} < 1$]

(31) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n+1}$

SINCE $\lim_{n\to\infty} \frac{n}{n+1} = 1 \ne 0$, THIS SERIES $\boxed{\text{DIVERGES}}$ BY THE DIVERGENCE TEST.

(SINCE $\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n}{n+1} \ne 0$, $\lim_{n\to\infty} a_n \ne 0$)

(33) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$

A) $\sum_{n=1}^{\infty} |a_n| = \sum_{n=1}^{\infty} \frac{100^n}{n!}$

SINCE $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{100^{n+1}}{(n+1)!} \cdot \frac{n!}{100^n} = \lim_{n\to\infty} \frac{100}{n+1} = 0 < 1$,

$\sum_{n=1}^{\infty} |a_n|$ CONVERGES BY THE RATIO TEST, SO $\sum_{n=1}^{\infty} a_n$ $\boxed{\text{CONVERGES ABSOLUTELY}}$.

---

(5) $\sum_{n=1}^{\infty} (-1)^n \frac{n}{n^2+1}$ $\boxed{\text{CONVERGES}}$ BY THE AST SINCE

1) $\lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{\frac{1}{n}}{1+\frac{1}{n^2}} = \frac{0}{1} = 0$ AND

2) $\frac{n}{n^2+1} \ge \frac{n+1}{(n+1)^2+1}$ FOR ALL n, SINCE

$\frac{n}{n^2+1} \ge \frac{n+1}{n^2+2n+2}$ IFF $n(n^2+2n+2) \ge (n^2+1)(n+1)$ IFF

$n^3 + 2n^2 + 2n \ge n^3 + n^2 + n + 1$ IFF $n^2 + n \ge 1$ IFF $n(n+1) \ge 1$ IFF $n \ge 1$

(OR SHOW THAT IF $f(x) = \frac{x}{x^2+1}$, THEN $f'(x) = \frac{(1-x)(1+x)}{(x^2+1)^2} < 0$ FOR $x > 1$

SO f IS DECREASING FOR $x \ge 1$.)

(IT IS CONDITIONALLY CONVERGENT.)