10. \[ \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{\ln n} \] since \[ \lim_{n \to \infty} \frac{1}{\ln n} = 0 \] and \[ \frac{1}{\ln n} \leq \frac{1}{\ln(n+1)} \] for \( n \geq 2 \).

This series converges by the AST.

20. \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \]

a) \[ \sum_{n=1}^{\infty} \frac{1}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \]

\[ Q = \lim_{n \to \infty} \frac{\ln n!}{\ln n} = \lim_{n \to \infty} (n!)^{\frac{1}{n}} \leq \lim_{n \to \infty} n^{\frac{1}{n}} \]

so \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \] diverges by the Ratio Test and therefore \[ \sum_{n=1}^{\infty} \frac{1}{n} \] diverges also.

26. \[ \sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^2} \]

a) \[ \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \leq \frac{1}{n^2} \]

for all \( n \), so \[ \sum_{n=1}^{\infty} \frac{|\sin n|}{n^2} \] converges by the Comparison Test.

And therefore \[ \sum_{n=1}^{\infty} \frac{(-1)^n \sin n}{n^2} \] converges absolutely.

27. \[ \sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n} \]

a) \[ \sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n} \]

compare to \[ \sum_{n=1}^{\infty} \frac{2\left(\frac{1}{3}\right)^n}{n+5^n} \]

which converges since it is a geometric series with \( |r| < 1 \): \[ \frac{2}{n+5^n} \leq \frac{2}{5^n} \] for all \( n \), so \[ \sum_{n=1}^{\infty} \frac{2^{n+1}}{n+5^n} \] converges by the Comparison Test and therefore \[ \sum_{n=1}^{\infty} \frac{(-2)^{n+1}}{n+5^n} \] converges absolutely.

28. \[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \]

a) \[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \] is constant and decreasing for \( x \geq 2 \), and

\[ \int_{2}^{\infty} \frac{dx}{x \ln x} = \lim_{n \to \infty} \int_{2}^{n} \frac{du}{u \ln u} = \lim_{n \to \infty} \left[ \ln(\ln u) \right]_{2}^{n} \]

\[ = \lim_{n \to \infty} \ln(\ln n) - \ln(\ln 2) = \infty \]

so \[ \sum_{n=2}^{\infty} \frac{1}{n \ln n} \] diverges by the Integral Test.

b) \[ \lim_{n \to \infty} \frac{1}{n \ln n} = 0 \] and \[ \frac{1}{n \ln n} \geq \frac{1}{(n+1) \ln(n+1)} \] for \( n \geq 2 \), so

\[ \sum_{n=2}^{\infty} \frac{(-1)^{n+1}}{n \ln n} \] converges by the AST and therefore it is conditionally convergent.
10.6 - (a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n (n!)^2 \cdot \gamma^n}{(2n+1)!} \]

\[ q = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{(n+1)!}{(2n+3)!} \cdot \frac{(2n+1)!}{(n)!^2 \cdot 3^n} \cdot \frac{3^n}{(n+1)!} = \lim_{n \to \infty} \frac{3(n+1)^n}{(n+3)(n+1)^n} = \lim_{n \to \infty} \frac{3}{n+6} = \frac{3}{6} < 1, \]

so \[ \sum_{n=1}^{\infty} |a_n| \] converges by the Ratio Test and therefore \[ \sum_{n=1}^{\infty} a_n \] converges absolutely.

10.5 - (a) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot 3^n} \]

\[ q = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{(n+1)^{n+1}}{n^n} \cdot \frac{3^n}{(n+1)! \cdot 3^{n+1}} = \lim_{n \to \infty} \frac{3}{n+1} \cdot (\frac{n+1}{n})^n = \frac{3}{2} < 1, \]

so \[ \sum_{n=1}^{\infty} |a_n| \] converges by the Ratio Test and therefore \[ \sum_{n=1}^{\infty} a_n \] converges absolutely.

4. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n \cdot \ln(n)^2} \]

\[ f(x) = \frac{1}{x \cdot \ln(x)^2} \text{ is continuous and decreasing for } x \geq 2, \]

\[ \int_2^{\infty} \frac{1}{x \cdot \ln(x)^2} \, dx = \lim_{t \to \infty} \left[ -\frac{1}{\ln(t)} \right]_2^T = \lim_{t \to \infty} \left[ -\frac{1}{\ln(t)} \right]_2^T = \lim_{t \to \infty} \left( -\frac{1}{\ln(T)} - (-\frac{1}{\ln(2)}) \right) = \frac{1}{\ln(2)}, \]

so \[ \sum_{n=2}^{\infty} |a_n| \] converges by the Integral Test and therefore \[ \sum_{n=2}^{\infty} a_n \] converges absolutely.

8. \[ \sum_{n=1}^{\infty} \frac{n^{n+1}}{(2n+1)!} \]

\[ q = \lim_{n \to \infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n \to \infty} \frac{(n+1)^{n+1}}{(2n+3)!} \cdot \frac{(2n+1)!}{n^{n+1} \cdot 2^{n+1}} = \lim_{n \to \infty} \frac{n^{n+1}}{2n+1} = \frac{n^{n+1}}{2n+1} \]

so \[ \sum_{n=1}^{\infty} |a_n| \] converges by the Root Test and therefore \[ \sum_{n=1}^{\infty} a_n \] converges absolutely.
3. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}} \]

\[ \lim_{n \to \infty} \frac{1}{\sqrt{n}} = \lim_{n \to \infty} \frac{1}{n^{1/2}} = \frac{1}{1} = 1 \neq 0, \]

so the series diverges by the Divergence Test.

4. \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} \]

A) \[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \]

Compare to \[ \sum_{n=1}^{\infty} \frac{1}{n^2} \], which diverges (Harmonic Series):

\[ \lim_{n \to \infty} \frac{1}{n(n+1)} = \lim_{n \to \infty} \frac{1}{n} = -\frac{1}{n} = \lim_{n \to \infty} \frac{n^2}{n^2} = \lim_{n \to \infty} \frac{n}{n^2} = 1 > 0, \]

so \[ \sum_{n=1}^{\infty} \frac{1}{n(n+1)} \] diverges by the LCT.

B) \[ \sum_{n=1}^{\infty} \frac{(-1)^n}{n(n+1)} \]

Converges by The AST since \[ \lim_{n \to \infty} \frac{1}{n(n+1)} = 0 \] and \[ \frac{1}{n(n+1)} \geq \frac{1}{(n+1)(n+2)} \] for all \( n \), so it converges conditionally.

5. \[ \sum_{n=2}^{\infty} \frac{\cos n}{n(n-1)} \]

A) \[ \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \]

\[ \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \]

Converges by comparison with \[ \sum_{n=2}^{\infty} \frac{1}{n^2} \] using the LCT since \[ \sum_{n=2}^{\infty} \frac{1}{n^2} \] converges (p-series with \( p > 1 \), with 1 term deleted) and

\[ \lim_{n \to \infty} \frac{\frac{1}{n(n-1)}}{\frac{1}{n^2}} = \lim_{n \to \infty} \frac{n^2}{n(n-1)} = \lim_{n \to \infty} \frac{n}{n-1} = 1 < \infty \]

So \[ \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \]

Converges by the Comparison Test (with \[ \sum_{n=2}^{\infty} \frac{1}{n(n-1)} \])

And therefore \[ \sum_{n=2}^{\infty} \frac{\cos n}{n(n-1)} \] converges absolutely.

\[ \left( \frac{\pi}{2} \right) \]

6. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1} \sin \frac{1}{n}}{n(n+1)} \]

\[ \lim_{n \to \infty} \frac{\sin \frac{1}{n}}{n} = \lim_{n \to \infty} \frac{1}{n} = \lim_{n \to \infty} \frac{\sin \frac{1}{n}}{n} = \lim_{n \to 0} \frac{\cos \frac{1}{n}}{n} = 4 \neq 0, \]

so the series diverges by the Divergence Test.

7. \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+8)^2} \]

A) \[ \sum_{n=1}^{\infty} \frac{n}{(n+8)^2} \]

Compare to \[ \sum_{n=1}^{\infty} \frac{1}{n} \], which diverges (Harmonic Series):

\[ \lim_{n \to \infty} \frac{n}{(n+8)^2} = \lim_{n \to \infty} \frac{n}{n+8} \]

\[ = \lim_{n \to \infty} \frac{n^2}{n+8} = \lim_{n \to \infty} \frac{n}{n+8} = \frac{1}{1} = 1 > 0, \]

so \[ \sum_{n=1}^{\infty} \frac{1}{n(n+8)} \] diverges by the LCT.

B) \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(n+8)^2} \]

Converges by The AST since

1) \[ \lim_{n \to \infty} \frac{n}{(n+8)^2} = \lim_{n \to \infty} \frac{n^2}{n+8} = \lim_{n \to \infty} \frac{1}{(n+8)^2} = 0 \]

2) \[ \frac{1}{(n+8)^2} \]

For all \( n \), so it converges.
(13) \[ \sum_{n=1}^{\infty} \frac{S^n \sin(n)}{n!} \]

a) \[ \sum_{n=1}^{\infty} \frac{|a_n|}{n!} = \sum_{n=1}^{\infty} \frac{S^n}{n!} \]

\[ = \sum_{n=1}^{\infty} \frac{S^n}{n!} \]

Converges by the Ratio Test since

\[ \lim_{n \to \infty} \frac{(S^{n+1})}{(n+1)!} \cdot \frac{n!}{S^n} = \lim_{n \to \infty} \frac{S}{n+1} = 0 < 1 \]

and \[ \frac{S^n |\sin(n)|}{n!} \leq \frac{S^n}{n!} \]

for all \( n \).

So \[ \sum_{n=1}^{\infty} |a_n| \]

Converges by the Comparison Test and therefore

\[ \sum_{n=1}^{\infty} a_n \]

Converges Absolutely.

(14) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \tan^{-1}\left(\frac{1}{n^2}\right) \]

a) \[ \sum_{n=1}^{\infty} \frac{|a_n|}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \]

Comparing to \( \sum_{n=1}^{\infty} \frac{1}{n^2} \), which converges (\( p \)-series, \( p > 1 \));

\[ \lim_{n \to \infty} \tan^{-1}\left(\frac{1}{n^2}\right) = \lim_{x \to 0} \frac{\tan^{-1}x}{x} = \lim_{x \to 0} \frac{\frac{1}{1+x^2}}{x} = 1 < \infty \]

(letting \( x = \frac{1}{n^2} \)),

so \[ \sum_{n=1}^{\infty} |a_n| \]

Converges by the Comparison Test and \[ \sum_{n=1}^{\infty} a_n \]

Converges Absolutely.

(15) \[ \sum_{n=1}^{\infty} (-1)^n \frac{\ln(n)}{n^2} \]

a) \[ \sum_{n=1}^{\infty} \frac{|a_n|}{n^2} = \sum_{n=1}^{\infty} \frac{\ln(n)}{n^2} \]

Comparing to \( \sum_{n=1}^{\infty} \frac{1}{n^{3/4}} \), which converges (\( p \)-series, \( p > 1 \));

\[ \ln n < \sqrt{n} \text{ for all } n \neq 1 \]

so \[ \ln(n) < \frac{\sqrt{n}}{n^{3/4}} \text{ for all } n \neq 1 \]

and thus \[ \sum_{n=1}^{\infty} |a_n| \]

Converges by the CT and \[ \sum_{n=1}^{\infty} a_n \]

Converges Absolutely.

(16) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{\pi}{2n}\right) \]

a) \[ \sum_{n=1}^{\infty} \frac{|a_n|}{\pi^{2n}} = \sum_{n=1}^{\infty} \frac{\sin\left(\frac{\pi}{2n}\right)}{\pi^{2n}} \]

Comparing to \( \sum_{n=1}^{\infty} \frac{1}{\pi^{2n}} \), which diverges (Mult. of the Harmonic Series);

\[ \lim_{n \to \infty} \sin\left(\frac{\pi}{2n}\right) \cdot \frac{1}{n^2} = \lim_{x \to 0} \frac{\sin x}{x} = 1 > 0 \]

(letting \( x = \frac{\pi}{2n} \)),

so \[ \sum_{n=1}^{\infty} |a_n| \]

Diverges by the LCT.

b) \[ \sum_{n=1}^{\infty} (-1)^{n+1} \sin\left(\frac{\pi}{2n}\right) \]

Converges by the AST since

1) \[ \lim_{n \to \infty} \sin\left(\frac{\pi}{2n}\right) = 0 \]

and 2) \[ \sin\left(\frac{\pi}{2(n+1)}\right) > \sin\left(\frac{\pi}{2n}\right) \]

for all \( n \geq 1 \)

Since \( \frac{\pi}{2n} \geq \frac{\pi}{2(n+1)} \text{ for all } n \) and \( f(x) = \sin x \) is increasing on \( \left[0, \frac{\pi}{2}\right] \),

so \[ \sum_{n=1}^{\infty} |a_n| \]

Converges Conditionally.

* (Notice that \( \sin\left(\frac{\pi}{2n}\right) > 0 \), since \( \frac{\pi}{2n} \) is in \( (0, \frac{\pi}{2}) \) for all \( n \).)