MATH 21C — Sec. 10.1 – EXAMPLES

A) PROVE THAT $\displaystyle\lim_{n\to\infty} \frac{2n+1}{n+3} = 2$.

(PREPARATION: $\left|\frac{2n+1}{n+3} - 2\right| = \left|\frac{2n+1}{n+3} - \frac{2(n+3)}{n+3}\right| = \left|\frac{2n+1-2n-6}{n+3}\right| = \left|\frac{-5}{n+3}\right| = \frac{5}{n+3} < \varepsilon$ IFF

$\frac{n+3}{5} > \frac{1}{\varepsilon}$ IFF $n+3 > \frac{5}{\varepsilon}$ IFF $n > \frac{5}{\varepsilon} - 3$)

PF LET $\varepsilon > 0$ BE GIVEN, AND LET N BE A POSITIVE INTEGER WITH $N \geq \frac{5}{\varepsilon} - 3$.

IF $n > N$, THEN $n > \frac{5}{\varepsilon} - 3 \Rightarrow n+3 > \frac{5}{\varepsilon} \Rightarrow \frac{1}{n+3} < \frac{\varepsilon}{5} \Rightarrow$

$\frac{5}{n+3} < \varepsilon \Rightarrow \left|\frac{-5}{n+3}\right| < \varepsilon \Rightarrow \left|\frac{2n+1}{n+3} - 2\right| < \varepsilon$.

REMARK WE COULD INSTEAD HAVE CHOSEN N TO BE AN INTEGER WITH $N \geq \frac{5}{\varepsilon}$, USING THE FACT THAT $\frac{5}{n+3} < \frac{5}{n}$.

B) PROVE THAT $\displaystyle\lim_{n\to\infty} \frac{n}{n^2+4} = 0$.

(PREPARATION: $\left|\frac{n}{n^2+4} - 0\right| = \frac{n}{n^2+4} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon$ IF $n > \frac{1}{\varepsilon}$)

PF LET $\varepsilon > 0$ BE GIVEN, AND LET N BE AN INTEGER WITH $N \geq \frac{1}{\varepsilon}$.

IF $n > N$, THEN $n > \frac{1}{\varepsilon} \Rightarrow \frac{1}{n} < \varepsilon \Rightarrow \frac{n}{n^2+4} < \frac{n}{n^2} = \frac{1}{n} < \varepsilon$

$\Rightarrow \left|\frac{n}{n^2+4} - 0\right| < \varepsilon$.

C) PROVE THAT $\displaystyle\lim_{n\to\infty} \frac{n^2-3}{n^2+1} = 1$.

(PREPARATION: $\left|\frac{n^2-3}{n^2+1} - 1\right| = \left|\frac{n^2-3}{n^2+1} - \frac{n^2+1}{n^2+1}\right| = \left|\frac{-4}{n^2+1}\right| = \frac{4}{n^2+1}$, AND

$\frac{4}{n^2+1} < \frac{4}{n^2} < \varepsilon$ IF $\frac{n^2}{4} > \frac{1}{\varepsilon}$ IFF $n^2 > \frac{4}{\varepsilon}$ IFF $n > \frac{2}{\sqrt{\varepsilon}}$)

PF LET $\varepsilon > 0$ BE GIVEN, AND LET N BE AN INTEGER WITH $N \geq \frac{2}{\sqrt{\varepsilon}}$.

IF $n > N$, THEN $n > \frac{2}{\sqrt{\varepsilon}} \Rightarrow n^2 > \frac{4}{\varepsilon} \Rightarrow \frac{n^2}{4} > \frac{1}{\varepsilon} \Rightarrow$

$\frac{4}{n^2} < \varepsilon \Rightarrow \frac{4}{n^2+1} < \frac{4}{n^2} < \varepsilon \Rightarrow \left|\frac{n^2-3}{n^2+1} - 1\right| < \varepsilon$.

REMARK WE COULD INSTEAD HAVE CHOSEN N TO BE A POSITIVE INTEGER WITH $N \geq \sqrt{\frac{4}{\varepsilon} - 1}$ IF $0 < \varepsilon \leq 4$, AND $N = 1$ IF $\varepsilon > 4$.