MATH 21C

Sec. 14.8 - Lagrange Multipliers with 2 Constraints

To find the extrema of $f(x,y,z)$ subject to the constraints

$g(x,y,z)=k$ and $h(x,y,z)=l$,

solve the system

$$\boxed{\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \\ g(x,y,z) &= k \\ h(x,y,z) &= l \end{aligned}}$$

(Here there are 2 Lagrange multipliers, λ and μ.)

Ex Find the extrema of $f(x,y,z) = x + 7y + 9z$ subject to the constraints $\underbrace{x+y+z=1}_{g(x,y,z)}$ and $\underbrace{y^2+z^2=4}_{h(x,y,z)}$

$1 = \lambda(1) + \mu(0)$ so $\lambda = 1$

$7 = \lambda(1) + \mu(2y)$ $\quad 6 = 2\mu y$ so $y = \frac{3}{\mu}$

$9 = \lambda(1) + \mu(2z)$ $\quad 8 = 2\mu z$ so $z = \frac{4}{\mu}$

Then $y^2+z^2=4 \Rightarrow \frac{9}{\mu^2} + \frac{16}{\mu^2} = 4 \Rightarrow \frac{25}{\mu^2} = 4 \Rightarrow \mu^2 = \frac{25}{4} \Rightarrow \mu = \pm\frac{5}{2}$

1) If $\mu = \frac{5}{2}$, $\quad y = \frac{3}{\mu} = \frac{6}{5}$ and $z = \frac{4}{\mu} = \frac{8}{5}$ and $x = 1-y-z = -\frac{9}{5}$

2) If $\mu = -\frac{5}{2}$, $\quad y = \frac{3}{\mu} = -\frac{6}{5}$ and $z = \frac{4}{\mu} = -\frac{8}{5}$ and $x = 1-y-z = \frac{19}{5}$

$f\left(-\frac{9}{5}, \frac{6}{5}, \frac{8}{5}\right) = -\frac{9}{5} + \frac{42}{5} + \frac{72}{5} = 21$ is the max. value

$f\left(\frac{19}{5}, -\frac{6}{5}, -\frac{8}{5}\right) = \frac{19}{5} - \frac{42}{5} - \frac{72}{5} = -19$ is the min. value

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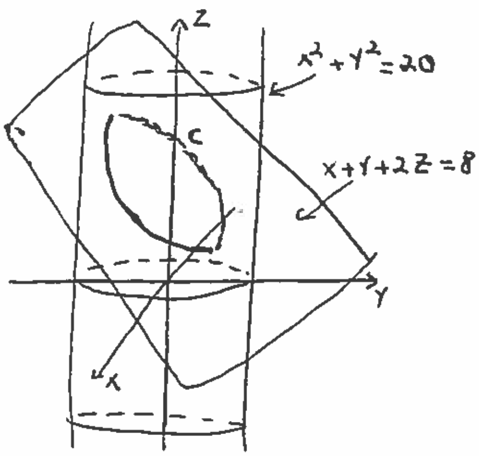
Remark In some cases, we can reduce these problems to finding the extrema of a function of 2 variables with 1 constraint, or even to finding the extrema of a function of 1 variable.

Ex In the problem above, $x = 1-y-z$; so we could find the extrema of $g(y,z) = 1 + 6y + 8z$ subject to $y^2+z^2=4$.

MATH 21C SEC. 14.8 – EXAMPLE (2 CONSTRAINTS)

FIND THE MAX. AND MIN. VALUE OF $f(x,y,z) = 4x + 3y + 4z$ ON THE INTERSECTION OF THE PLANE $x + y + 2z = 8$ AND THE CYLINDER $x^2 + y^2 = 20$.



LET $g(x,y,z) = x + y + 2z$ AND $h(x,y,z) = x^2 + y^2$;

SOLVING
$$\boxed{\begin{aligned} f_x &= \lambda g_x + \mu h_x \\ f_y &= \lambda g_y + \mu h_y \\ f_z &= \lambda g_z + \mu h_z \\ g(x,y,z) &= 8 \\ h(x,y,z) &= 20 \end{aligned}}$$

GIVES
$$4 = \lambda \cdot 1 + \mu \cdot 2x$$
$$3 = \lambda \cdot 1 + \mu \cdot 2y$$
$$4 = \lambda \cdot 2 + \mu \cdot 0 \implies \underline{\lambda = 2},$$

SO
$$4 = 2 + \mu \cdot 2x \implies 2 = \mu \cdot 2x \stackrel{1)}{\implies} \underline{1 = \mu \cdot x}$$
$$3 = 2 + \mu \cdot 2y \stackrel{2)}{\implies} \underline{1 = \mu \cdot 2y}$$

MULTIPLYING 1) BY y GIVES $\underline{y = \mu x y}$, AND

MULTIPLYING 2) BY x GIVES $\underline{x = 2\mu x y}$.

THEREFORE $\underline{x = 2y}$, SO SUBSTITUTING INTO $x^2 + y^2 = 20$

GIVES $4y^2 + y^2 = 20$, SO $5y^2 = 20$, $y^2 = 4$, AND $\underline{y = \pm 2}$

a) IF $y = 2$, $x = 4$ AND $z = \frac{1}{2}(8 - 4 - 2) = 1$:

$\boxed{f(4,2,1) = \underline{26} \text{ IS THE MAX. VALUE}}$

b) IF $y = -2$, $x = -4$ AND $z = \frac{1}{2}(8 - (-4) - (-2)) = 7$:

$\boxed{f(-4,-2,7) = \underline{6} \text{ IS THE MIN. VALUE}}$
