MATH 21C

ANSWERS – TEST 3

① $f(x,y) = x^2 + 3xy - 2y^2$ AT $P(2,3)$

$\vec{\nabla} f = \langle 2x+3y,\ 3x-4y\rangle$ so $\vec{\nabla} f(P) = \langle 13,-6\rangle$ $\qquad \vec{u} = \dfrac{\vec{v}}{|\vec{v}|} = \dfrac{\langle 3,4\rangle}{5} = \langle \tfrac{3}{5}, \tfrac{4}{5}\rangle$

so $D_{\vec{u}} f(P) = \vec{\nabla} f(P)\cdot \vec{u} = \langle 13,-6\rangle \cdot \langle \tfrac{3}{5},\tfrac{4}{5}\rangle = \frac{39}{5} - \frac{24}{5} = \boxed{3}$

② $2x^2 - 5xz + 2yz + y^2 = 13$ AT $P(2,3,1)$

$\vec{\nabla} F = \langle 4x-5z,\ 2z+2y,\ -5x+2y\rangle$ $\qquad \vec{\nabla} F(P) = \langle 3,8,-4\rangle = \vec{n}$

$3x + 8y - 4z = d$ WHERE $d = 6+24-4 = 26$ GIVES $\boxed{3x+8y-4z=26}$

③ EXTREMA OF $f(x,y,z) = 3x - y - 2z$ ON ELLIPSOID $\underbrace{3(x-5)^2 + 2y^2 + (z+3)^2}_{g(x,y,z)} = 120$:

$\vec{\nabla} f = \lambda(\vec{\nabla} g)$:

$3 = \lambda\cdot 6(x-5)$	$\frac{1}{\lambda} = 2(x-5)$	$2(x-5) = -4y$	$x-5 = -2y$	$x = -2y+5$
$-1 = \lambda \cdot 4y$	$\frac{1}{\lambda} = -4y$			
$-2 = \lambda\cdot 2(z+3)$	$\frac{1}{\lambda} = -(z+3)$	$-(z+3) = -4y$	$z+3 = 4y$	$z = 4y-3$

SUBSTITUTING INTO THE CONSTRAINT GIVES

$3(-2y)^2 + 2y^2 + (4y)^2 = 120,\quad 3(4y^2) + 2y^2 + 16y^2 = 120,\quad 30y^2 = 120,\quad y^2 = 4,\quad y = \pm 2$

$y=2$: $f(1,2,5) = -9$ IS THE MIN.

$y=-2$: $f(9,-2,-11) = 51$ IS THE MAX.

④ A) $f_x = 3x^2 + y^2 - 12x = 0$

$f_y = 2xy - 2y = 0 \longrightarrow 2y(x-1) = 0$ $\quad$ 1) $y=0$ OR 2) $x=1$

1) IF $y=0$, $\quad 3x^2 - 12x = 0,\ 3x(x-4) = 0,\ x=0$ OR $x=4$

2) IF $x=1$, $\quad 3 + y^2 - 12 = 0,\ y^2 = 9,\ y = \pm 3$

CRITICAL POINTS: $\boxed{(0,0),\ (4,0),\ (1,3),\ (1,-3)}$

B) $f_{xx} = 6x - 12$

$f_{xy} = 2y$

$f_{yy} = 2x - 2$

	f_{xx}	f_{xy}	f_{yy}	D
(0,0)	−12	0	−2	24
(4,0)	12	0	6	72
(1,3)	−6	6	0	−36
(1,−3)	−6	−6	0	−36

LOCAL MAX, AT (0,0)
LOCAL MIN, AT (4,0)
SADDLE PT. AT (1,3)
SADDLE PT. AT (1,−3)

⑤ $w = f\left(\frac{4t}{u}, t^2u^3\right)$ WHERE $f_x = xy^2$ AND $f_y = x^2y + 5y$ $\quad$ LET $x = \frac{4t}{u}$ AND $y = t^2u^3$:

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\cdot\frac{\partial y}{\partial u} = f_x\cdot\frac{\partial x}{\partial u} + f_y\cdot\frac{\partial y}{\partial u}$$

$$= (xy^2)\left(-\frac{4t}{u^2}\right) + (x^2y + 5y)(3t^2u^2)$$

$$= \left(\frac{4t}{u}\right)(t^4u^6)\left(-\frac{4t}{u^2}\right) + \left(\frac{16t^2}{u^2}\cdot t^2u^3 + 5t^2u^3\right)(3t^2u^2)$$

$$= -16t^6u^3 + 48t^6u^3 + 15t^4u^5$$

$$= \boxed{32t^6u^3 + 15t^4u^5}$$

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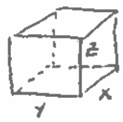
⑥ $f(x,y,z) = \dfrac{x^2}{y} - 2yz$, $P = (4, 2, -3)$

$\vec{\nabla} f = \left\langle \dfrac{2x}{y}, -\dfrac{x^2}{y^2} - 2z, -2y \right\rangle$ so $\vec{\nabla} f(P) = \langle 4, 2, -4 \rangle$

A) $|\vec{\nabla} f(P)| = \sqrt{4^2 + 2^2 + 4^2} = \boxed{6}$ IS THE MAX. DIRECTIONAL DERIVATIVE OF f AT P.

B) $\vec{u} = -\dfrac{\vec{\nabla} f(P)}{|\vec{\nabla} f(P)|} = -\dfrac{\langle 4, 2, -4\rangle}{6} = \boxed{\left\langle -\frac{2}{3}, -\frac{1}{3}, \frac{2}{3} \right\rangle}$

⑦ 

(OPEN TOP)

1) MINIMIZE $S = xy + 2yz + 2xz$

2) $V = xyz = 500$, so $z = \dfrac{500}{xy}$ AND

$S = xy + \dfrac{1000}{x} + \dfrac{1000}{y}$

3) $S_x = y - \dfrac{1000}{x^2} = 0$ $\quad y = \dfrac{1000}{x^2}$ $\quad$ so $x^2 y = 1000 = xy^2$, AND $x = y$ (SINCE $x, y \neq 0$)

$S_y = x - \dfrac{1000}{y^2} = 0$ $\quad x = \dfrac{1000}{y^2}$

THEN $x^3 = 1000$, so $\boxed{x = 10 \text{ cm}, \; y = 10 \text{ cm}, \; z = 5 \text{ cm}}$ $\leftarrow \dfrac{500}{10(10)}$

⑧ POINTS ON $5x^2 - 6xy + 5y^2 = 64$ CLOSEST TO ORIGIN:

MINIMIZE $f(x,y) = d^2 = x^2 + y^2$ TO GET $\vec{\nabla} f = \lambda (\vec{\nabla} g)$ OR

$2x = \lambda(10x - 6y)$ $\qquad \dfrac{1}{\lambda} = \dfrac{10x - 6y}{2x} = 5 - \dfrac{3y}{x}$ $\quad$ so $\dfrac{3y}{x} = \dfrac{3x}{y}$, $\quad 3y^2 = 3x^2$,

$2y = \lambda(-6x + 10y)$ $\qquad \dfrac{1}{\lambda} = \dfrac{10y - 6x}{2y} = 5 - \dfrac{3x}{y}$ $\qquad y^2 = x^2$, $\; y = \pm x$

1) IF $y = x$, $5x^2 - 6x^2 + 5x^2 = 64$, $4x^2 = 64$, $x^2 = 16$, $x = \pm 4$: $(4,4)$ AND $(-4,-4)$

2) IF $y = -x$, $5x^2 + 6x^2 + 5x^2 = 64$, $16x^2 = 64$, $x^2 = 4$, $x = \pm 2$: $(2,-2)$ AND $(-2,2)$

SINCE $f(4,4) = f(-4,-4) = 32$ AND $f(2,-2) = f(-2,2) = 8$,

$\boxed{(2,-2) \text{ AND } (-2,2)}$ ARE THE CLOSEST POINTS TO $(0,0)$.

⑨ $\underbrace{x^2 + 5y^2 + z^2}_{f(x,y,z)} = \dfrac{25}{4}$ AND $\underbrace{x^2 - xz^2 + 6yz}_{g(x,y,z)} = 5$ AT $P(2, \frac{1}{2}, 1)$

$\vec{\nabla} f = \langle 2x, 10y, 2z \rangle$ so $\vec{n}_1 = \vec{\nabla} f(P) = \langle 4, 5, 2 \rangle$

$\vec{\nabla} g = \langle 2x - z^2, 6z, -2xz + 6y \rangle$ so $\vec{n}_2 = \vec{\nabla} g(P) = \langle 3, 6, -1 \rangle$

$\vec{a} = \vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & 5 & 2 \\ 3 & 6 & -1 \end{vmatrix} = \langle -17, 10, 9 \rangle$ IS A DIRECTION VECTOR FOR THE TANGENT LINE TO THE CURVE AT P:

$$\boxed{\begin{aligned} x &= 2 - 17t \\ y &= \tfrac{1}{2} + 10t \\ z &= 1 + 9t \end{aligned}}$$, t in $\mathbb{R}$

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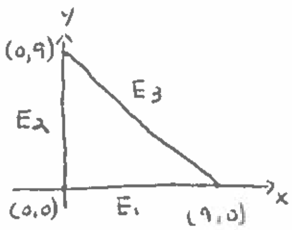
(10)

$V = \pi r^2 h \qquad r: 5 \to 5.11 \qquad h: 10 \to 9.8$

$$dV = V_r \,\Delta r + V_h \,\Delta h = 2\pi r h \,\Delta r + \pi r^2 \,\Delta h$$
$$= 2\pi(5)(10)(.11) + \pi(5^2)(-.2)$$
$$= 11\pi - 5\pi = \boxed{6\pi \text{ cm}^3}$$

(11)



$f(x,y) = x^2 + y^2 - xy - 4x - y$

1) $f_x = 2x - y - 4 = 0$ $\qquad 2x - y = 4$ $\qquad$ (+) $\quad 4x - 2y = 8$

$f_y = 2y - x - 1 = 0$ $\qquad -x + 2y = 1$ $\qquad\quad -x + 2y = 1$

$3x = 9$

so $x = 3$ and $y = 2$ gives the only critical point in the interior of the region: $\boxed{(3,2)}$

2) on E_1, $g(x) = f(x,0) = x^2 - 4x$ so
$g'(x) = 2x - 4 = 0$ for $x = 2$: $\boxed{(2,0)}$

3) on E_2, $h(y) = f(0,y) = y^2 - y$ so
$h'(y) = 2y - 1 = 0$ for $y = \frac{1}{2}$: $\boxed{(0,\tfrac{1}{2})}$

4) on E_3, $k(x) = x^2 + (9-x)^2 - x(9-x) - 4x - (9-x)$
$= 3x^2 - 30x + 72$ so
$k'(x) = 6x - 30 = 0$ for $x = 5$ and $y = 4$: $\boxed{(5,4)}$

(As noted in the problem, the vertices $(0,0)$, $(9,0)$, and $(0,9)$ are the other points to consider.)