2) \textbf{Show that every bounded decreasing sequence converges.}\n
\textbf{PF} Let \((s_n)\) be a bounded decreasing sequence, and let \(i = \inf E\) where \(E = \{s_n : n \in \mathbb{N}\}\) and \(s_n < i + \varepsilon\) for some \(n \in \mathbb{N}\).

Then \(i - \varepsilon < s_n < i + \varepsilon\) for \(n \geq N\), so \(n \geq N \Rightarrow |s_n - i| < \varepsilon\).

Therefore, \(\lim_{n \to \infty} s_n = i = \inf E\).

3) \textbf{Let } \(S\) \textbf{ be a nonempty bounded subset of } \(\mathbb{R}\) \textbf{ such that } \(\sup S \neq S\). \textbf{Prove that there is a sequence } \((s_n)\) \textbf{ with } \(s_n \in S\) \textbf{ for all } \(n\) \textbf{ and } \(\lim s_n = \sup S\).\n
\textbf{PF} \(T = \sup S\).

For each \(n \in \mathbb{N}\), \(T - \frac{1}{n} < T\) is not an upper bound for \(S\).

So \(T - \frac{1}{n} < s_n\) for some \(s_n \in S\) where \(s_n < T\) since \(T = \sup S\) and \(T \neq S\).

Since \(T - \frac{1}{n} < s_n < T\) for all \(n \in \mathbb{N}\) and \(\lim (T - \frac{1}{n}) = T\) and \(\lim T = T\), \(\lim s_n = T\) by the Squeeze Theorem.

4) \(S_1 = 1\) and \(S_{n+1} = \left(\frac{n}{n+1}\right)S_n^2\) for \(n \geq 1\).

a) \(S_2 = \frac{1}{2}, S_3 = \frac{1}{6}, S_4 = \frac{1}{18}\)

b) 1) \(S_n \leq 1\) for all \(n \in \mathbb{N}\) by induction:

i) This is true for \(n = 1\), since \(S_1 = 1 \leq 1\).

ii) Assume that \(S_n \leq 1\) for some \(n \in \mathbb{N}\), then \(S_{n+1} = \left(\frac{n}{n+1}\right)S_n^2 < S_n^2 \leq 1^2 = 1\).

2) \(S_{n+1} \leq S_n\) for all \(n \in \mathbb{N}\) since \(S_{n+1} = \left(\frac{n}{n+1}\right)S_n^2 < S_n^2 \leq 1, S_0 = S_0\).

3) \(S_n > 0\) for all \(n \in \mathbb{N}\) by induction:

i) This is true for \(n = 1\), since \(S_1 = 1 > 0\).

ii) Assume that \(S_n > 0\) for some \(n \in \mathbb{N}\), then \(S_{n+1} = \left(\frac{n}{n+1}\right)S_n^2 > 0\).

Since \((S_n)\) is decreasing by 2) and bounded below by 3), it converges by the Monotone Convergence Theorem.

c) If \(\lim S_n = S\), then \(\lim S_{n+1} = \lim_{n \to \infty} \left(\frac{n}{n+1}\right)S_n^2 = 0 = 1 \cdot S^2 = S^2\).

And \(S^2 = 3\) implies that \(S(S-1) = 0\) so \(S = 0\) or \(S = 1\).

Since \(S \leq \frac{1}{x}\) for \(x \geq 2\), \(S \leq \frac{1}{2}\) and therefore \(S = 0\).
1) Show that if \( \lim_{n \to \infty} S_n = 0 \), then \( \lim_{n \to \infty} \frac{1}{S_n} = 0 \).

**Proof (PF):** Let \( \varepsilon > 0 \) be given. Since \( \lim_{n \to \infty} S_n = 0 \), there is a positive integer \( N \) such that if \( n \geq N \), then \( S_n > \frac{1}{\varepsilon} \).

Therefore, if \( n \geq N \), then \( \left| \frac{1}{S_n} - 0 \right| = \frac{1}{S_n} < \varepsilon \), so \( \lim_{n \to \infty} \frac{1}{S_n} = 0 \).

b) Show that if \( \lim_{n \to \infty} S_n = 0 \) and \( S_n \geq 0 \) for all \( n \), then \( \lim_{n \to \infty} \frac{1}{S_n} = \infty \).

**Proof (PF):** Let \( K > 0 \) be given. Since \( \lim_{n \to \infty} S_n = 0 \), there is a positive integer \( N \) such that if \( n \geq N \), then \( |S_n - 0| < \frac{1}{K} \) so \( S_n < \frac{1}{K} \) (since \( S_n \geq 0 \) for all \( n \)).

Then \( \frac{1}{S_n} > K \) for \( n \geq N \), so \( \lim_{n \to \infty} \frac{1}{S_n} = \infty \).

2) Prove that \( \lim_{n \to \infty} T^n = 0 \) if \( r > 1 \).

**Proof (PF):** Let \( T = \frac{1}{r} \), so \( 0 < T < 1 \). Then \( \lim_{n \to \infty} T^n = 0 \) (by Th. 9.9b), so \( \lim_{n \to \infty} T^n = \lim_{n \to \infty} \frac{1}{T^n} = 0 \) by (1b).

3) PF Let \( K > 0 \), and let \( N \) be in \( \mathbb{N} \) with \( N \geq \frac{L.N.K}{L.N.} \).

If \( n \geq N \), then \( n \geq \frac{L.N.K}{L.N.} \Rightarrow n.L.N.K \geq L.N.K \) (since \( L.N. > 0 \)), therefore \( L.N.K \geq L.N.K \), so \( n \geq K \) if \( n \geq N \). Thus \( \lim_{n \to \infty} n^n = \infty \).

4) PF Let \( K > 0 \), and let \( \lambda = r-1 > 0 \). If \( N \) be in \( \mathbb{N} \) with \( N \geq \frac{K-1}{\lambda} \), then \( n \geq N \Rightarrow n \geq \frac{K-1}{\lambda} \Rightarrow n > K-1 \Rightarrow 1 + n > K \), since \((1 + \lambda)^n \geq 1 + n\lambda \) by Bernoulli's inequality, \( n \geq N \Rightarrow (1 + \lambda)^n \geq K \Rightarrow n^n \geq K \), so \( \lim_{n \to \infty} n^n = \infty \).

3) Prove that if \( \lim_{n \to \infty} S_n = 0 \), then \( \lim_{n \to \infty} |15n| = 151 \).

**Proof (PF):** Let \( \varepsilon > 0 \) be given.

Since \( \lim_{n \to \infty} S_n = 0 \), there is an \( N \) in \( \mathbb{N} \) such that if \( n \geq N \), then \( |S_n - 3| < \varepsilon \).

Therefore \( n \geq N \Rightarrow |15n - 151| < 15 \varepsilon | < \varepsilon \) by the Triangle Inequality.

So \( \lim_{n \to \infty} |15n| = 151 \).

Claim: If \( (S_n) \) does not have a convergent subsequence, then \( \lim_{n \to \infty} |15n| = \infty \).

PF (Of the Contrapositive):
Suppose that \( \lim_{n \to \infty} |15n| \neq \infty \); then there is a \( \lambda > 0 \) such that for every \( N \in \mathbb{N} \), there is an \( n \geq N \) such that \( |15n| < \lambda \). Therefore \( |15n| < \lambda \) for infinitely many values of \( n \), so \( (S_n) \) has a subsequence \( (S_{n_k}) \) with \( |15n_k| < \lambda \) for all \( \lambda \in \mathbb{N} \).

Since \( (S_{n_k}) \) is a bounded sequence, it has a convergent subsequence \( (S_{n_{k_1}}, S_{n_{k_2}}, S_{n_{k_3}}, \ldots) \) by the Bolzano-Weierstrass Theorem.