0. \( E = \{ x \in \mathbb{Q} : x > 0 \text{ and } x^2 < 2 \} \)

E is bounded above by 2, since \( \forall x \in E \Rightarrow x^2 < 2 \Rightarrow x < \sqrt{2} \) and \( E \neq \emptyset \) since \( 1 \in E \).

1. If \( x \in \mathbb{Q} \), then \( x^2 \neq 2 \) since we showed that \( \sqrt{2} \) is irrational.

2. If \( x^2 > 2 \), then \( x^4 - 2 = \frac{4x^4 + 8x^2 + 4}{x^2 + 4} - 2 = \frac{2x^4 - 4}{(x^2 + 4)} \Rightarrow x^2 > 2 \Rightarrow x^2 - 2 > 0 \Rightarrow x^2 > 2 \).

3. \( x^2 > 2 \Rightarrow 4x^2 > 8x^2 + 8 > (2x + 2)^2 > 2x + 4x + 4 \Rightarrow (2x + 2)^2 > 2 \Rightarrow x^2 = \left( \frac{2x + 2}{2} \right)^2 > 2 \).

4. Since \( \gamma = \frac{2x + 2}{2} \), \( \gamma > 0 \) (since \( x > 0 \)) and

\( x < \gamma \Rightarrow 2x + 2 < x^2 + 4 \Rightarrow 5 \Rightarrow \gamma < 5 \).

5. If \( x \in E \), then \( x < \gamma \) and \( x^2 < 2 \), \( s \in E \) and therefore \( \gamma \) is an upper bound for \( E \). Since \( \gamma < 5 \), this contradicts the assumption that \( \gamma = \sup E \).

6. \( x^2 - 2 = \frac{4x^4 + 8x^2 + 4}{x^2 + 4} - 2 = \frac{2x^4 - 4}{(x^2 + 4)} \Rightarrow x^2 - 2 < 0 \Rightarrow x^2 < 2 \).

7. If \( x^2 < 2 \), then \( 4x^2 < 8x^2 + 8 > (2x + 2)^2 > 2x + 4x + 4 \Rightarrow (2x + 2)^2 < 2x + 4x + 4 \Rightarrow (2x + 2)^2 < 2 \Rightarrow x^2 = \left( \frac{2x + 2}{2} \right)^2 < 2 \).

8. Since \( x^2 < 2 \), \( x^2 + 4 < x^2 + 4 \Rightarrow (2x + 2)^2 < 2x + 4x + 4 \Rightarrow \gamma < 5 \).

9. Since \( x^2 < 2 \), \( x^2 + 4 < x^2 + 4 \Rightarrow \gamma < 5 \).

10. \( \gamma = \frac{2x + 2}{2} \Rightarrow x^2 < 2 \Rightarrow \gamma < 5 \).

11. Since \( \gamma > 0 \) and \( x^2 < 2 \), \( x \in E \) with \( \gamma > 3 \), and

this contradicts the assumption that \( \gamma \) is an upper bound for \( E \).
S = \{ x \in \mathbb{R} : x > 0 \text{ and } x^2 < 3 \}

a) S \neq \emptyset \text{ since } 1 \in S, \text{ and } 2 \text{ is an upper bound for } S \text{ since } x \in S \Rightarrow x^2 < 3 < 4 \Rightarrow x < 2.

b) If \( t^2 < 3 \), by the Archimedes Property, there is an \( m \in \mathbb{N} \) with \( \frac{1}{m} < \frac{3 - t^2}{2t} \), so \( \frac{2t + 1}{m} < t^2 + \frac{3 - t^2}{2t} = \frac{t^2 + 3 - t^2}{2t} = 3 \).

Thus \( t + \frac{1}{m} \in S \) with \( t + \frac{1}{m} > t \), and this gives a contradiction since \( t \) is an upper bound for \( S \).

c) If \( t^2 \geq 3 \), by the Archimedes Property, there is an \( m \in \mathbb{N} \) with \( \frac{1}{m} < \frac{t^2 - 3}{2t} \), so \( \frac{2t}{m} < t^2 - 3 \).

If \( x \in S \), then \( x > 0 \) with \( x^2 < 3 \leq (t - \frac{1}{m})^2 \),

so \( x < t - \frac{1}{m} \) and therefore \( t - \frac{1}{m} \) is an upper bound for \( S \).

Since \( t - \frac{1}{m} < t \) and \( t = \sup S \), this gives a contradiction.

\[ \textbf{3. Show that } \mathbb{R} \text{ is uncountable using the Nested Intervals Property.} \]

\[ \textbf{PF (by contradiction):} \]

Suppose instead that \( \mathbb{R} \) is countable, so there is a bijection \( f : \mathbb{N} \to \mathbb{R} \).

Then \( \mathbb{R} = \{ x_1, x_2, x_3, \ldots \} \) where \( x_n = f(n) \) for each \( n \in \mathbb{N} \).

Now construct a sequence of nested intervals \( I_n = [a_n, b_n] \) as follows:

1) Let \( I_1 = [a_1, b_1] \) be any interval with \( x_1 \notin I_1 \).

2) Let \( I_2 = [a_2, b_2] \) be an interval satisfying \( I_1 \supseteq I_2 \) and \( x_2 \notin I_2 \).

(For example, if \( x_2 \notin I_1 \), let \( I_2 = I_1 \); and if \( x_2 \in I_1 \), divide \( I_1 \) into 3 equal subintervals and let \( I_2 \) be a subinterval with \( x_2 \notin I_2 \).)

Continuing in this manner (by induction),

we get an infinite sequence of nested (closed) bounded intervals

\[ I_1 \supseteq I_2 \supseteq I_3 \supseteq \ldots \supseteq I_n \supseteq \ldots \text{ with } x_n \notin I_n \text{ for each } n \in \mathbb{N}, \]

by the Nested Intervals Property, there is a number \( x \) with \( x \notin I_n \) for all \( n \), and this gives a contradiction since \( x \notin \mathbb{N} \) for all \( n \) and therefore \( x \notin \mathbb{R} \).

\[ \textbf{4. } F = \{ \frac{P(x)}{Q(x)} : P(x) \text{ and } Q(x) \text{ are polynomials} \} \text{ and} \]

\[ f > g \iff f - g = \frac{P(x)}{Q(x)} \text{ and } \frac{a}{b} > 0 \text{ where } a \text{ and } b \text{ are the leading coefficients of } P(x) \text{ and } Q(x), \text{ respectively,} \]

\[ F \text{ does not satisfy the Archimedean Property,} \]

since \( x \in F \) satisfies \( x > n \) for all \( n \in \mathbb{N} \) because

\[ x - n = x - n \]

where \( \frac{a}{b} = \frac{1}{1} > 0. \]
Solutions - Problem Sheet 3

5) \( S = \{ \sqrt{5} : r \in \mathbb{Q} \} \)

a) Let \((x, y)\) be an open interval. Since \(Q\) is dense in \(\mathbb{R}\), there is an \(r \in Q\) with \(r \in (\frac{x}{\sqrt{5}}, \frac{y}{\sqrt{5}})\), so
\[
\frac{x}{\sqrt{5}} < r < \frac{y}{\sqrt{5}} \implies x < r\sqrt{5} < y,
\]
therefore \(r\sqrt{5} \in (x, y)\) with \(\sqrt{5} \in S\), so \(S\) is dense in \(\mathbb{R}\).

b) As shown in class previously, \(r \in Q\) with \(r \neq 0\) and \(x \notin Q \implies r\sqrt{5} \notin Q\), so all the non-zero elements of \(S\) are irrational.

If \((x, y)\) is an open interval, it contains a \(x \in S\) by part a). If \(S \neq 0\), then \(S\) is irrational; and if \(S = 0\), then there is an \(x \in S\) which is in \((0, y)\) by part a), so then \(\sqrt{5}\) is irrational. Therefore \((x, y)\) contains an irrational number in either case, so the irrational numbers are dense in \(\mathbb{R}\).

Remark: Problem #12 in Sec. 4 gave another way to show this.

6) Show that if \(x \in \mathbb{R}\) and \(m, n \in \mathbb{Z}\) with \(m \leq x < m + 1\) and \(n \leq x < n + 1\), then \(m = n\).

Proof (by contradiction)

Suppose instead that \(m \neq n\), so either \(m < n\) or \(n < m\).

We can assume (without loss of generality) that \(m < n\), then \(m \leq n - 1\) since \(m, n \in \mathbb{Z}\); so
\[
x < m + 1 \leq n \leq x \implies x < x,
\]
which gives a contradiction.

Therefore \(m = n\).
a) Even though #1 is to show that Q does not satisfy the completeness axiom and #2 is to show that \( \sqrt{3} \) exists in \( \mathbb{R} \), both problems involve similar ideas:

In #1, we have to get a number \( t \) closer to \( \sqrt{2} \) than 3 (with \( t \in \mathbb{Q} \)), and in #2, we have to find a number closer to \( \sqrt{3} \) than \( t \).

b) In #1, there is more than one way to find \( t \in \mathbb{Q} \) which is closer to \( \sqrt{2} \) than 3:

1. 

\[
\frac{\sqrt{2} - 3}{\sqrt{2} + 3}
\]

If \( \sqrt{2} < t \), then

\[
\sqrt{2} - 3 = \frac{\sqrt{2} + 3}{\sqrt{2} + 3} \cdot \frac{\sqrt{2} - 3}{\sqrt{2} + 3} > \frac{\sqrt{2} - 3}{\sqrt{2} + 3},
\]

so

\[
\sqrt{2} = 3 + \frac{2 - 3^2}{2 + 3} = \frac{3 + 2}{2 + 3}
\]

satisfies \( \sqrt{2} < t \) when \( \sqrt{2} \).

(if \( \sqrt{2} > 3 \), a similar argument gives the same expression for \( t \).)

2. a) If \( \sqrt{2} > 3 \), Newton's method with \( f(x) = x^2 - 2 \) gives

\[
y = x^2 - 2
\]

\[
\frac{\sqrt{2} - 3}{\sqrt{2} + 3}
\]

b) If \( \sqrt{2} < 3 \), Newton's method with \( f(x) = x - \frac{2}{x} \) gives

\[
y = x - \frac{2}{x}
\]

\[
\frac{\sqrt{2} - 3}{\sqrt{2} + 3}
\]

3. Since \( Q \) satisfies the Archimedean Property, we could use the same ideas as in #2 to find

\[
\gamma = 3 + \frac{1}{n} \quad \text{with} \quad (3 + \frac{1}{n})^2 < 2 \quad \text{if} \quad 3^2 < 2,
\]

and

\[
\gamma = 3 - \frac{1}{n} \quad \text{with} \quad (3 - \frac{1}{n})^2 > 2 \quad \text{if} \quad 3^2 > 2.
\]
3) a) \( \lim_{n \to \infty} \frac{n}{n+1} = 1 \)

f) \( \lim_{n \to \infty} 2^{1/n} = 1 \)

i) \( \lim_{n \to \infty} \frac{(-1)^n}{n} = 0 \)

k) \( \lim_{n \to \infty} \frac{9n^2 - 18}{6n + 18} \) does not exist, so the sequence diverges.

p) \( \lim_{n \to \infty} \frac{2^{n+1} + 5}{2^n - 7} = \lim_{n \to \infty} \frac{2 + \frac{5}{2^n}}{1 - \frac{7}{2^n}} = \frac{2}{1} = 2 \)

q) \( \lim_{n \to \infty} \frac{3n}{n^2} = 0 \)

r) \( \lim_{n \to \infty} \left( 1 + \frac{1}{n} \right)^2 = 1^2 = 1 \)

s) \( \lim_{n \to \infty} \frac{4n^3 + 3}{3n^3 - 2} = \frac{4}{3} \)

t) \( \lim_{n \to \infty} \frac{6n + 4}{9n^2 + 7} = 0 \)

4) a) \( \text{Let } x_n = \frac{\sqrt{n}}{n} \). Then \( x_n = \frac{1}{n\sqrt{n}} \) is irrational for each \( n \in \mathbb{N} \)

since \( \frac{1}{n} \in \mathbb{Q} \), \( \frac{1}{n} \neq 0 \), and \( \sqrt{2} \notin \mathbb{Q} \); with \( \lim_{n \to \infty} x_n = 0 \).

[As another example, let \( x_n = \sqrt{1 + \frac{1}{n}} \) for all \( n \).]

Then \( x_n \) is irrational for each \( n \in \mathbb{N} \), using proof by contradiction, but \( \lim_{n \to \infty} x_n = 1 \).]