A.5 - Prove that \( \sqrt{2} \) is irrational.

PF (by contradiction)

Suppose instead that \( \sqrt{2} \) is rational, so \( \sqrt{2} = \frac{m}{n} \) where \( m, n \in \mathbb{Z} \), \( n \neq 0 \), and \( \frac{m}{n} \) is reduced to lowest terms.

Then \( 2 = \frac{m^2}{n^2} \), so \( m^2 = 2n^2 \) and therefore \( m^2 \) is even.

Therefore \( m \) is even, so \( m = 2k \) for some \( k \in \mathbb{Z} \) and \( 4k^2 = 2n^2 \).

Then \( n^2 = 2k^2 \), so \( n^2 \) is even and therefore \( n \) is even.

The fact that \( m \) and \( n \) are both even contradicts our assumption that \( \frac{m}{n} \) was in reduced form, so \( \sqrt{2} \) must be irrational.

[Remark: We proved in class that \( m \) is even whenever \( m^2 \) is even by proving the contrapositive, and later we'll prove \( \sqrt{2} \) is irrational using the WOP.]

3. Prove that there are infinitely many primes.

PF 1 (by contradiction)

Suppose instead that there are only finitely many primes \( p_1, \ldots, p_n \), and let \( q = p_1 \cdots p_n + 1 \). Since \( q \) is an integer and \( q > 1 \), \( q \) has a prime factor \( q \neq p_i \) for some \( i \) with \( 1 \leq i \leq n \).

Suppose \( q \mid (p_1 \cdots p_n) \), \( p_i \mid (p_1 \cdots p_n) \), and \( q \mid (q - p_i) \). Thus \( q \mid 1 \), and this contradicts the fact that \( 1 \) has no prime factors.

So there must be infinitely many primes.

PF 2 (by contradiction)

Suppose instead that there are only finitely many primes, and let \( p \) be the largest prime. If \( n \neq p \), then \( n \) is an integer greater than 1, so \( n \) has a prime factor \( q \neq p \).

Since \( q < p \), \( q \mid p! \) so \( q \mid 1 \) and \( q \mid 1 \) and therefore \( q \mid (p - 1) \). Thus \( q \mid 1 \), and this contradicts the fact that \( 1 \) has no prime factors.

So there must be infinitely many primes.

[Remarks: PF 1 is the proof given by Euclid; and we will show the fact used in both proofs that any integer greater than 1 has at least one prime factor. We will also show the fact used in PF 2 that any finite set has a largest element.]

P.3 - Prove that if \( x \) is irrational and \( \gamma \) is rational, then \( x + \gamma \) is irrational.

PF (by contradiction)

Suppose instead that \( x \) is irrational, \( \gamma \) is rational, and \( x + \gamma \) is rational.

Then \( x = (x + \gamma) - \gamma \) would be rational, which gives a contradiction.

Therefore \( x \) is irrational and \( \gamma \) is rational.

Then \( x + \gamma \) must be irrational.