3.7) **False:** Let $S_n = n$ and $T_n = -n$; then $\{S_n\}$ and $\{S_n + T_n\}$ diverge, but $\{S_n + T_n\}$ converges.

**False:** Let $S_n = (-1)^n$ and $T_n = (-1)^{n+1}$; then $\{S_n\}$ and $\{S_n + T_n\}$ diverge, but $\{S_n + T_n\}$ converges.

**True:** This follows from Th. 2.15, since $T_n = (S_n + T_n) - S_n$.

**False:** Let $S_n = \frac{1}{n}$ and $T_n = n$; then $\{S_n\}$ and $\{S_n + T_n\}$ converge, but $\{T_n\}$ diverges.

**False:** Let $S_n = \frac{1}{n}$; then $\{S_n\}$ converges, but $\{\frac{1}{S_n}\}$ diverges.

**True:** This follows from Th. 2.16, since $(S_n)^2 = S_n \cdot S_n$.

**False:** Let $S_n = (-1)^n$; then $\{(S_n)^2\}$ converges, but $\{S_n\}$ diverges.

3.9) **Let** $S_n = \frac{1.3.5 \ldots (2n-1)}{1.4.6 \ldots (2n)}$.

Since $\frac{S_{n+1}}{S_n} = \frac{1.3.5 \ldots (2n-1)(2n+1)}{1.4.6 \ldots (2n)(2n+2)} \cdot \frac{2.4.6 \ldots (2n)}{1.3.5 \ldots (2n-1)} = \frac{2n+1}{2n+2} < 1$ for all $n$,

$S_{n+1} < S_n$ for all $n$ and therefore $\{S_n\}$ is decreasing.

Since $S_n > 0$ for all $n$, $\{S_n\}$ is bounded below by 0, so it converges by the Monotone Convergence Theorem.
3.11 - 4) Show that \( Z \) is a cluster point of \( \{S_n\} \) if there is a subsequence \( \{S_{n_k}\} \) converging to \( Z \).

**Proof:** Suppose \( Z \) is a cluster point of \( \{S_n\} \).

First choose \( S_{n_1} \) in \((Z - \varepsilon, Z + \varepsilon)\). Since there are infinitely many terms in \((Z - \frac{1}{n_1}, Z + \frac{1}{n_1})\), next choose \( S_{n_2} \) in \((Z - \frac{1}{n_1}, Z + \frac{1}{n_1})\) with \( n_2 > n_1 \).

Continuing in this manner, if \( S_{n_k} \) is in \((Z - \frac{1}{n_k}, Z + \frac{1}{n_k})\), we can choose \( S_{n_{k+1}} \) with \( n_{k+1} > n_k \) and \( S_{n_{k+1}} \) in \((Z - \frac{1}{n_{k+1}}, Z + \frac{1}{n_{k+1}})\). By induction, this gives a subsequence \( \{S_{n_k}\} \) with \( S_{n_k} \) in \((Z - \frac{1}{n_k}, Z + \frac{1}{n_k})\) for all \( k \in \mathbb{N} \).

Given \( \varepsilon > 0 \), let \( K \) be an integer with \( K > \frac{1}{\varepsilon} \).

If \( k > K \), then \( Z - \frac{1}{n_k} < \varepsilon \Rightarrow Z - \frac{1}{n_k} < \varepsilon \). Therefore, \( \lim_{n \to \infty} S_{n_k} = Z \).

4) If \( \{S_n\} \) converges to \( Z \), then for every \( \varepsilon > 0 \) there is an integer \( K \) such that \( |S_n - Z| < \varepsilon \) if \( k > K \). Therefore, there are infinitely many terms in \((Z - \varepsilon, Z + \varepsilon)\), so \( Z \) is a cluster point for \( \{S_n\} \).

**Remark:** This problem shows that the cluster points of a sequence are the same as its subsequential limits.

4.13 - 4) Show that \( \limsup n a_n = \lim \inf n a_n \).

**Proof:** Let \( S \) be the set of subsequential limits of \( \{a_n\} \), so \( S \) is the set of subsequential limits of \( \{-a_n\} \).

Then \( \limsup (-a_n) = \sup S = -\lim \inf a_n \) (using #7 in 1.6).

And \( \liminf a_n = \limsup (-a_n) = \limsup (-\lim \inf a_n) = -\lim \inf a_n \) (using #7 in 1.6).

4) Show that \( \limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n \) if \( \{a_n\} \) and \( \{b_n\} \) are bounded.

**Proof:** Let \( T_n = \{a_n: n \geq N\}, \ T_n' = \{b_n: n \geq N\} \), and \( T_n'' = \{a_n + b_n: n \geq N\} \) for each \( N \).

And let \( V_n = \sup T_n, \ V_n' = \sup T_n', \) and \( V_n'' = \sup T_n'' \) for each \( N \).

If \( n \geq N \), then \( a_n + b_n \leq V_n + V_n' \), so \( V_n + V_n' \) is an upper bound for \( T_n'' \).

Then \( \lim V_n'' \leq \lim V_n + \lim V_n' \),

So \( \limsup (a_n + b_n) \leq \limsup a_n + \limsup b_n \).

5) If we let \( a_n = (-1)^n \) and \( b_n = (-1)^n \),

Then \( a_n + b_n = 0 \) for all \( n \geq 0 \),

\( \limsup (a_n + b_n) = 0 \) while \( \lim \sup a_n + \lim \sup b_n = 1 + 1 = 2 \).

(Therefore the inequality is strict in this case.)

**Remark:** This result holds more generally, except in the case where \( \limsup a_n = \infty \) and \( \limsup b_n = -\infty \) or vice versa, if we adopt the conventions that \( x + (-\infty) = -\infty \) and \( x + \infty = \infty \) for any \( x \in \mathbb{R} \).