4. Let $a, b, c \in \mathbb{Z}$. Show that if $a^2 + b^2 = c^2$, then $a$ or $b$ is even.

**PF (By Contradiction)**

Suppose instead that $a^2 + b^2 = c^2$ and $a$ and $b$ are both odd, then $a = 2k + 1$ and $b = 2l + 1$ for some integers $k$ and $l$.

So $c^2 = a^2 + b^2 = (2k + 1)^2 + (2l + 1)^2 = 4k^2 + 4k + 1 + 4l^2 + 4l + 1 = 4k^2 + 4k + 4l^2 + 4l + 2 = 2(2k^2 + 2l^2 + 2k + 2l + 1)$

Therefore, $c^2$ is even, so $c$ is even and thus $c = 2m$ for some $m \in \mathbb{Z}$.

Then $4m^2 = c^2 = 2(2k^2 + 2l^2 + 2k + 2l + 1)$, so $2m^2 = k^2 + 2k + l^2 + 2l + 1$.

However, this is impossible, since $2m^2$ is even and $k^2 + 2k + l^2 + 2l + 1 = 2(k^2 + k + l^2 + l) + 1$ is odd.

Therefore, if $a^2 + b^2 = c^2$, then either $a$ is even or $b$ is even.

3. Let $m \in \mathbb{Z}$. Show that if $m^3$ is even, then $m$ is even.

**PF (Of the Contrapositive)**

If $m$ is even, then $m = 2k + 1$ for some $k \in \mathbb{Z}$.

Then $m^3 = (2k + 1)^3 = 8k^3 + 12k^2 + 6k + 1 = 2(4k^3 + 6k^2 + 3k) + 1$ where $4k^3 + 6k^2 + 3k \in \mathbb{Z}$, so $m^3$ is odd.

4. Show that if $x$ is irrational, then $5x$ is irrational.

**PF (Of the Contrapositive)**

If $5x$ is rational, then $x = \frac{5x}{5}$ is rational.

5. Prove or disprove: if $n \in \mathbb{Z}$ and $121n$ and $101n$, then $1201n$.

**False**: if $n = 60$, then $120160$ and $10160$ but $120 \nmid 60$.

6. Prove or disprove: if $a, b \in \mathbb{Z}$ and $6 \mid ab$, then $6 \mid a$ or $6 \mid b$.

**False**: if $a = 2$ and $b = 3$, then $6 \mid a$ but $6 \nmid a$ and $6 \nmid b$.

(or use $a = 4$ and $b = 9$, say)
1. c) $\forall x \in \mathbb{R}, x^2 \geq 0$

   The square of every real number is nonnegative. (T)

2. c) $\exists x \in \mathbb{R}, x^2 < 0$ (F)

3. d) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 1$

   For every real number $x$ and for every real number $y$, $x + y = 1$. (F)
   (equivalently, The sum of any two real numbers is 1.)

4. d) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \neq 1$ (T)

5. e) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$

   For every real number $x$, there is a real number $y$ such that $x + y = 1$. (T)

6. e) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y \neq 1$ (F)

7. f) $\exists x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y = 1$

   There is a real number $x$ such that for every real number $y$, $x + y = 1$. (F)

8. f) $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y \neq 1$ (T)

9. g) $\exists x \in \mathbb{R}, \exists y \in \mathbb{R}, x + y = 1$

   There exists a real number $x$ and a real number $y$ such that $x + y = 1$. (T)
   (equivalently, There exist two real numbers whose sum is 1.)

10. g) $\forall x \in \mathbb{R}, \forall y \in \mathbb{R}, x + y \neq 1$ (F)