\( 1 \cdot 13 \cdot 24 \cdot 35 \cdot 46 \cdot 57 = 729 \cdot 162 \cdot 256 \cdot 361 \cdot 484 \cdot 625 = 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 = 7 ! \cdot 13 ! \)

\( \text{Claim:} \quad 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad \text{for all } n \in \mathbb{N}, \)

\( \text{Proof:} \quad \text{This is true for } n = 1, \text{ since } 1 = 1^2. \)

\( a) \quad \text{Assume that } 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad \text{for some } n \in \mathbb{N}. \)

\( \text{Then } \quad 1 + 3 + 5 + \cdots + (2n - 1) + (2n + 1) = n^2 + (2n + 1) = (n + 1)^2, \)

so the formula is true for \( n + 1. \)

Therefore, \( 1 + 3 + 5 + \cdots + (2n - 1) = n^2 \quad \text{for all } n \in \mathbb{N} \text{ by the PMI.} \)

\( \text{Show that } 7^n - 4^n \text{ is divisible by } 3 \quad \text{for all } n \in \mathbb{N}. \)

\( \text{Proof:} \quad \text{This is true for } n = 1, \text{ since } 3| (7^1 - 4^1) \text{ because } 3 \mid 3. \)

\( a) \quad \text{Assume that } 7^n - 4^n \text{ is divisible by } 3 \quad \text{for some } n \in \mathbb{N}. \)

\( \text{Then } \quad 7^n - 4^n = 3k \quad \text{for some } k \in \mathbb{Z}, \text{ so } 7^n = 4^n + 3k \text{ and } \)

\( 7^{n+1} - 4^{n+1} = 7 (7^n) - 4 (4^n) = 7 (4^n + 3k) - 4 (4^n) = 21k + 3 (4^n) = 3 (7^n + 4^n); \)

so \( 7^{n+1} - 4^{n+1} \text{ is divisible by } 3. \)

Therefore, \( 7^n - 4^n \text{ is divisible by } 3 \quad \text{for all } n \in \mathbb{N} \text{ by the PMI.} \)

\( \text{Show that if } x > 0, \text{ then } (1 + x)^n \geq 1 + nx \quad \text{for all } n \in \mathbb{N}. \)

\( \text{Proof:} \quad \text{This is true for } n = 1, \text{ since } 1 + x \geq 1 + x. \)

\( a) \quad \text{Assume that } (1 + x)^n \geq 1 + nx \quad \text{for some } n \in \mathbb{N}. \)

\( \text{Then } \quad (1 + x)^{n+1} = (1 + x) (1 + x)^n \geq (1 + x) (1 + nx) \quad \text{(since } 1 + x > 0) \)

\( = 1 + x + nx + nx^2 \)

\( = 1 + (n+1)x + nx^2 \geq 1 + (n+1)x \quad \text{since } nx^2 \geq 0. \)

Therefore, \( (1 + x)^n \geq 1 + nx \quad \text{for all } n \in \mathbb{N} \text{ by the PMI.} \)

\( \text{Remark:} \quad \text{This inequality, which we'll use later on, is called Barrow's inequality;} \)

\( \text{and the proof shows that it is valid for } x > -1. \)

\( \text{Show that } 1^3 + 2^3 + 3^3 + \cdots + n^3 = \left( \frac{n (n+1)}{2} \right)^2 \quad \text{for all } n \in \mathbb{N}. \)

\( \text{Proof:} \quad \text{Since } 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n(n+1)^2}{4} \text{ (as shown on p. A14-A15),} \)

\( \text{it is enough to show that } 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2 (n+1)^2}{4} \text{ for all } n \in \mathbb{N}. \)

\( a) \quad \text{This is true for } n = 1, \text{ since } 1^3 = 1 = \frac{(1(1+1))}{4}. \)

\( a) \quad \text{Assume that } 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2 (n+1)^2}{4}. \quad \text{for some } n \in \mathbb{N}. \)

\( \text{Then } \quad \left[ 1^3 + 2^3 + 3^3 + \cdots + n^3 \right] + (n+1)^3 = \frac{n^2 (n+1)^2 + (n+1)^3}{4} = \left( \frac{n^2}{4} + \frac{1}{4} \right) \quad \text{(n+1)^2} \]

\( \text{and } \quad \text{so the formula is valid for } n+1. \)

Therefore, \( 1^3 + 2^3 + 3^3 + \cdots + n^3 = \frac{n^2 (n+1)^2}{4} = \left( \frac{1+2+3+\cdots+n+1}{2} \right)^2 \quad \text{for all } n \in \mathbb{N} \text{ by the PMI.} \)
3. Every integer \( n > 1 \) has a prime factor.

**Proof (by contradiction)**

Suppose instead that this assertion is false, so there is a set \( T \) containing all integers greater than 1 which do not have a prime factor. Since \( T \) is nonempty, \( T \) has a least element \( m \) by the Well-Ordering Principle. But \( m \) is not a prime (since \( m > 1 \)) and \( m \) is not in \( T \) (since otherwise \( m \) would be a prime factor of itself).

Therefore, there is an integer \( k \) such that \( 1 < k < m \) and \( k \) is not in \( T \). Since \( k < m \), \( k \) is the last element of \( T \), \( k \not\in T \). Therefore, \( k \) has a prime factor \( p \) (since \( k > 1 \)). Since \( p \) and \( k \) are integers, \( pk \) is in \( T \). But this contradicts the fact that \( m \) is the last element of \( T \). Therefore, every integer \( n > 1 \) has a prime factor.

**Remark**: We used this result in A.5-2, and this type of proof is called a proof by minimal counterexample.

4. Every integer \( n > 1 \) can be written as a product of primes.

**Proof**

1. The assertion is true for \( n = 2 \), since 2 is prime.
2. Let \( n \) be an integer with \( \gcd(n, 2) = 1 \), and assume that the statement is true for all integers \( k \) with \( 2 < k < n \).
   a. If \( n + 1 \) is prime, then the statement is true for \( n + 1 \).
   b. If \( n + 1 \) is not prime, then \( n + 1 = kl \), where \( k \) and \( l \) are integers in \( \{2, 3, \ldots, n\} \). Therefore, \( k \) and \( l \) can be written as products of primes by the induction hypothesis, so \( n + 1 \) is also a product of primes.

Thus, in either case, the statement is true for \( n + 1 \), therefore every integer \( n > 1 \) can be written as a product of primes by strong induction.

5. a) Suppose that \( 6a + 9b + 20c = 43 \) for some non-negative integers \( a, b, \) and \( c \).
   i) If \( a \leq 0 \), \( 6a + 9b = 43 \) has no solution since \( 3| \) (6a + 9b) but \( 3 \nmid 43 \).
   ii) If \( a \geq 1 \), \( 6a + 9b = 43 \) has no solution since \( 3| \) (6a + 9b) but \( 3 \nmid 43 \).
   iii) If \( a = 0 \), \( 6a + 9b = 43 \) has no solution since \( 6a + 9b \geq 6 \) unless \( a = 0 \).

Therefore, 43 is not a McNugget number.

b) Every integer \( n \geq 44 \) is a McNugget number.

**Proof**

1. The integers \( 44, 45, \ldots, 49 \) are McNugget numbers since they can be written as \( 44 = 6(1) + 20(1) \), \( 45 = 9(5) \), \( 46 = 6(1) + 20(2) \), \( 47 = 9(3) + 20(1) \), \( 48 = 6(8) \), and \( 49 = 9(1) + 20(2) \).

Let \( n \) be an integer with \( n \geq 44 \), and assume that all integers \( k \) with \( 44 \leq k \leq n \) are McNugget numbers.

Since \( 44 \leq n - 5 \leq n \), \( n - 5 \) is a McNugget number and therefore \( n - 5 = 6a + 9b + 20c \). Then \( n + 1 = 6(a + 1) + 9b + 20c \), so \( n + 1 \) is a McNugget number.

Therefore, every integer \( n \geq 44 \) is a McNugget number by strong induction.

**Remark**: Notice that we verified 6 cases in the base step, since 6 was the smallest size order of McNuggets allowed.