1. a) \( n \) does not satisfy A3, A4, and M4.

b) \( 2 \) does not satisfy M4.

2. Prove that \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3} \) for all \( n \in \mathbb{N} \).

Proof: Since \( 1 \cdot 2 = 2 = \frac{1 \cdot 2 \cdot 3}{3} \), the statement is true for \( n = 1 \).

a) Assume that \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3} \) for some \( n \in \mathbb{N} \).

Then \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) + (n+1) \cdot (n+2) \)

\[ = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2) \left( \frac{n+3}{3} \right) \]

\[ = \frac{n(n+1)(n+2) + 3(n+2)(n+3)}{3} \]

Therefore, \( 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots + n \cdot (n+1) = \frac{n(n+1)(n+2)}{3} \) for all \( n \in \mathbb{N} \) by the PMI.

3. Prove that \( 5^n - 4n - 1 \) is divisible by 16 for all \( n \in \mathbb{N} \).

Proof: Since \( 5^1 - 4 - 1 = 0 \) and 0 is divisible by 16, the statement is true for \( n = 1 \).

a) Assume that \( 5^n - 4n - 1 \) is divisible by 16 for some \( n \in \mathbb{N} \).

So \( 5^n - 4n - 1 = 16k \) for some \( k \in \mathbb{Z} \). Then \( 5^n = 16k + 4n + 1 \), so

\[ 5^{n+1} - 4(n+1) - 1 = 5(5^n) - 4n - 5 = 5(16k + 4n + 1) - 4n - 5 \]

\[ = 80k + 16n + 20 \quad \text{where} \quad 80k + 20 \in \mathbb{Z} \]

Therefore, \( 5^{n+1} - 4(n+1) - 1 \) is divisible by 16.

4. Prove that \( 2^n > (n+1)^2 \) for all integers \( n \geq 6 \).

Proof: Since \( 2^6 = 64 > 49 = 7^2 \), the statement is true for \( n = 6 \).

a) Assume that \( 2^n > (n+1)^2 \) for some integer \( n \geq 6 \).

Then \( 2^{n+1} = 2(2^n) > 2(n+1)^2 = 2(n^2 + 2n + 1) = 2n^2 + 4n + 2 \)

\[ = n^2 + (n^2 + 2n + 2) > n^2 + 2n + 2 = (n+1)^2 \]

so the inequality is valid for \( n+1 \).

Therefore, \( 2^n > (n+1)^2 \) for all integers \( n \geq 6 \) by the (generalized) PMI.