Let \((x, d)\) be a metric space, and let \(A \subseteq \mathbb{X}\).

**DEF** The closure of \(A\), denoted by \(\overline{A}\), is the intersection of all closed sets containing \(A\).

**REMARK**
1. \(\overline{A}\) is the smallest closed set containing \(A\), and
2. \(A\) is closed iff \(A = \overline{A}\).

**PROP** \(x \in \overline{A}\) iff \(B(x, \delta) \cap A \neq \emptyset\) for every \(\delta > 0\).

**PF**
- If \(x \notin \overline{A}\), then \(x \in A^C\) where \(A^C\) is open (since \(\overline{A}\) is closed).
- Then \(B(x, \delta) \subseteq A^C \subseteq A^C\) for some \(\delta > 0\), so \(B(x, \delta) \cap A\neq \emptyset\).
- \(\Rightarrow\) \(B(x, \delta) \cap A = \emptyset\) for some \(\delta > 0\), then \(A \subseteq B^C\) where \(B = B(x, \delta)\).
- Since \(B^C\) is a closed set containing \(A\), \(x \notin \overline{A}\) since \(A \subseteq B^C\) and \(x \notin B^C\).

**DEF** The interior of \(A\), denoted by \(A^0\), is defined by
\[A^0 = \{x \in X : B(x, \delta) \subseteq A \text{ for some } \delta > 0\}^0\]

**EX**
- \(A = [0, 2, 5]\) in \(\mathbb{R}\), \(A^0 = (0, 5)\).

**PROP** \(A^0\) is open.

**PF**
- Let \(x \in A^0\), so \(B(x, \delta) \subseteq A\) for some \(\delta > 0\). If \(y \in B(x, \delta')\), then since \(B(x, \delta)\) is open, \(B(y, \delta') \subseteq B(x, \delta) \subseteq A\) for some \(\delta' > 0\).
- Therefore \(y \in A^0\), so \(B(x, \delta) \subseteq A^0\). Thus \(A^0\) is open.

**REMARK**
1. \(A^0\) is the largest open set contained in \(A\), and
2. \(A\) is open iff \(A = A^0\).

**Sequences**

**DEF** A sequence \((x_n)\) in a metric space \((X, d)\) converges to \(x \in X\) iff for every \(\varepsilon > 0\) there is an \(N\in \mathbb{N}\) such that if \(n \geq N\), then \(x_n \in B(x, \varepsilon)\).

**REMARK** Notice that \(\lim_{n \to \infty} x_n = x\) iff \(\lim_{n \to \infty} d(x_n, x) = 0\).

**PROP** Let \(E \subseteq X\) be a closed set in a metric space \((X, d)\).
- If \(\{x_n\}\) is a sequence in \(E\) and \(\lim_{n \to \infty} x_n = x\), then \(x \in E\).

**PF** (by contradiction)
- If \(x \notin E\), then \(x \in E^C\) where \(E^C\) is open, so \(x \in B(x, \delta) \subseteq E^C\) for some \(\delta > 0\).
- Since \(\lim_{n \to \infty} x_n = x\), there exists \(N\in \mathbb{N}\) such that \(x_n \in E^C\) for all \(n \geq N\).
- This gives \(\lim_{n \to \infty} x_n \notin E\) since \(x \notin E\).

**PROP** If \(x \in \overline{A}\), then there is a sequence \((\alpha_n)\) in \(A\) which converges to \(x\).

**PF** Since \(x \in \overline{A}\), \(B(x, \frac{1}{n}) \cap A = \emptyset\) for each \(n \in \mathbb{N}\). If we let \(\alpha_n \in B(x, \frac{1}{n}) \cap A\) for each \(n\), then \(\lim_{n \to \infty} \alpha_n = x\) since \(d(\alpha_n, x) < \frac{1}{n}\).

**DEF** A point \(x \in X\) is a limit point of \(A\) iff \(B(x, \delta)\) contains an element of \(A\) other than \(x\) for every \(\delta > 0\).

**REMARK** Limit points are also called accumulation points or cluster points.

**PROP** A point \(x \in X\) is a limit point of \(A\) iff there is a sequence \((\alpha_n)\) in \(A\) such that \(\lim_{n \to \infty} \alpha_n = x\) and \(\alpha_n \neq x\) for all \(n \in \mathbb{N}\).

**PF**
- If \(A \subseteq X\) and \(L\) is the set of limit points of \(A\), then \(A = \overline{A} = A\cup L\).

**Cor** A set \(A\) is closed iff it contains all its limit points.

**DEF** If \(x \notin \overline{A}\) and \(x\) is not a limit point of \(A\), then \(x\) is called an isolated point of \(A\).

**DEF** A set \(A\) is perfect if it is closed and contains no isolated points.